Deliberation and the Wisdom of Crowds

Franz Dietrich
Paris School of Economics & CNRS

Kai Spiekermann
London School of Economics

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Abstract

Does pre-voting group deliberation increase majority competence? To address this question, we develop a probabilistic model of opinion formation and deliberation. Two new jury theorems, one pre-deliberation and one post-deliberation, suggest that deliberation is beneficial. Successful deliberation mitigates three voting failures: (1) overcounting widespread evidence, (2) neglecting evidential inequality, and (3) neglecting evidential complementarity. Simulations and theoretic arguments confirm this. But there are five systematic exceptions where deliberation reduces majority competence, always by increasing failure (1). Our analysis recommends deliberation that is 'participatory', 'even', but possibly 'unequal', i.e., that involves substantive sharing, privileges no evidences, but possibly privileges some persons.

1 Introduction: Deliberation and Voting

Does group deliberation improve group decisions? Many scholars of deliberation assert that it does, though others have warned that deliberation can fall into epistemic traps. Since the formal understanding of the epistemic merits of deliberation is at an early and disjointed stage, it is hard to assess who is right. We present a formal analysis of *deliberation as sharing and absorbing evidence.* We interpret 'sharing', 'absorbing' and 'evidence' broadly and therefore account for deliberation in its full breadth: as an exchange of empirical facts, but also arguments, perspectives, or other reasons. Our analysis provides a clearer understanding of when and how pre-voting deliberation is beneficial.

We assume that votes express judgments about what is socially correct, adopting an epistemic conception of democracy. Under this epistemic paradigm, voting is supposed to produce outcomes that best track the truth in response to the total evidence dispersed across voters. This can fail to happen, for at least three reasons:

- *Failure 1: Overcounting widespread evidence.* Evidence held by more voters has exaggerated influence, by affecting more votes.
- *Failure 2: Neglecting evidential inequality.* Voters have the same weight, despite their unequally strong total evidence.
- *Failure 3: Neglecting evidential complementarity.* Information obtainable after combining different evidences dispersed across voters is undercounted, because few or no voters access all these evidences simultaneously.

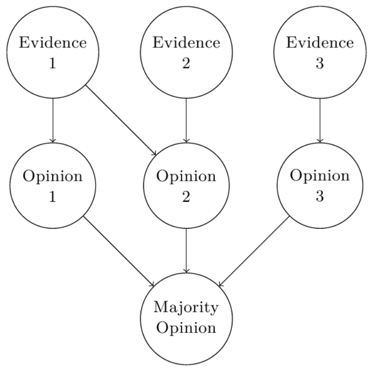

Figure 1: A simple example

All three failures stem from bad management of available but dispersed evidences. Figure 1 gives a stylised example with three voters and three evidences. Failure 1 arises because evidence 1 is overcounted; it affects two votes while evidences 2 and 3 each affect only one vote. Failure 2 arises because voter 2 has stronger total evidence.[1] Failure 3 arises, for instance, if evidences 1 and 3 are complementary, because no voter has them both. For example, evidences 1 and 3 could be arguments that are uninformative in isolation but highly informative in combination.

The hope is that deliberation can improve the use of evidence. Specifically:

- Deliberation could reduce Failure 1 by increasing the spread of previously private or almost private evidences.
- Deliberation could reduce Failure 2 by letting voters with initially weak total evidence accumulate evidence.
- Deliberation could reduce Failure 3 by letting voters collect evidences from others and then recognize and use evidential complementarities.

But are these conjectures correct? We introduce a formal model of deliberation, and, for the first time, prove jury theorems that address the effect of deliberation on voting outcomes. We then analyse each failure, partly through simulations. Our jury theorems and failure analysis largely confirm the optimistic take on deliberation. We present a typology of beneficial and harmful deliberation, allowing us to state more precisely which caveats apply to the thesis that deliberation promotes the 'wisdom of crowds'.

This paper is in 8 sections. After giving an account of existing approaches towards pre-voting deliberation in Section 2, Section 3 develops our new formal model of opinion formation and deliberation. Section 4 then presents two jury theorems that set an upper bound to collective competence, while suggesting that this upper bound is easier to reach post-deliberation. We also decompose the group's 'competence gap' into two gaps, one that deliberation can potentially close and another one that an increase of the group size can potentially close. This is followed by exploratory simulations in Section 5, suggesting that deliberation is much better at reducing Failure 2 than at reducing Failure 1, and supporting a recommendation for 'participatory' and 'even' deliberation. After generalising the framework in Section 6, we can address Failure 3 in Section 7, where we argue that deliberation robustly helps against Failure 3. Section 8 offers concluding considerations.

[1] Assuming her two evidences are stronger in total than evidence 1 and than evidence 3.

2 Deliberation and Voting in Context

A large interdisciplinary literature addresses the interaction of deliberation and voting. Three perspectives dominate: the social-choice-theoretic, game-theoretic, and normative-democratic perspective. We now describe these perspectives and explain how our own approach relates to them.

A first approach to deliberation comes from social choice theory, especially its epistemic branch. There is a long tradition of thinking about the epistemic effects of deliberation (Aristotle 1988 [350BC]; Condorcet 1785; Rousseau 1972 [1762]). Condorcet, a founder of social choice theory, was very much engaged with its epistemic aspect; not only did he interpret preferences epistemically as judgments of social betterness (McLean and Hewitt 1994, p. 38), he also proved the first of many jury theorems (e.g., Grofman et al. 1983, Ladha 1992, List and Goodin 2001, Dietrich and Spiekermann 2013; for a review see Dietrich and Spiekermann 2021). Unfortunately, deliberation has ambiguous effects on the assumptions of traditional jury theorems, potentially promoting voter competence while undermining voter independence. Although more recent jury theorems escape the concern that deliberation might undermine voter independence (e.g., Dietrich and Spiekermann 2013), no jury theorem addresses deliberation effects on (majority) outcomes. Our deliberation-specific jury theorems will aim to fill this gap. For other social-choice-theoretic takes on the epistemic virtues of deliberation, see Betz (2013), Perote-Pena and Piggins (2015), Goodin and Spiekermann (2018), Hartmann and Rafiee Rad (2016, 2019), and Hoek and Bradley (2022). Of course, pre-voting deliberation also serves non-epistemic purposes, such as enabling stable collective preferences (Dryzek and List 2003; Rafiee Rad and Roy 2021).

The game-theoretic literature interprets deliberation and voting as strategic interactions. Voters strategize about what and when to communicate, and then strategize about how to vote. One insight of this literature is that, even if all share the same epistemic preferences, incentives for strategic manipulation can arise in deliberation and voting, depending on the environment (e.g., Coughlan 2000; Austen-Smith and Feddersen 2006; Gerardi and Yariv 2007). While we see great value in the game-theoretic approach, especially for repeated interactions in small deliberative venues, we here model deliberation in a macroscopic rather than game-theoretic way, focusing on the structural properties of information flow rather than the micro-foundations of behaviour. We choose a macroscopic model because of two objectives: achieving parsimony and permitting several psychological interpretations and hypotheses. For instance, limited sharing and absorbing of sources could represent either conscious choices or unsuccessful attempts or hard inabilities; and deliberators could be instrumentally or intrinsically motivated, have stable or variable preferences, reason strategically or not, be fully rational or use simple heuristics, anticipate all contingencies or fail to imagine them before they occur during deliberation. Relatedly, deliberators could acquire new information or, more fundamentally, refine their awareness and concepts through which they perceive or interpret the situation. Game-theoretic models need to commit on all these issues, often in stylised and specific ways[2] that seem in tension with thinking in democratic theory about

[2]For instance, the game tree describing all contingencies is known to all players. This implies that only information can grow, not awareness. Some interesting modifications to game theory have been proposed to capture growing awareness, but they have not made their way into the mainstream as they involve unorthodox departures and complexities in the very notions of game and equilibrium (e.g.,

the cognitive and motivational structure of deliberation (e.g., Cohen 1996; Gutmann and Thompson 1996). Still, Landa and Meirowitz (2009) show how game theory can inform normative deliberative theory, by working out which institutional proposals are strategically stable, for example. The relative advantages of game-theoretic and non-game-theoretic approaches to modelling deliberation were already contrasted by Fearon (1998). We wish to emphasise two semi-game-theoretic models of deliberation: Chung and Duggan's (2020) model of myopic discussion, constructive discussion and debate, and Ding and Pivato's (2021) model of deliberation as a process of information disclosure. Our approach is related to theirs in its emphasis on information transmission, but is macroscopic rather than game-theoretic.

Finally, political theory, and normative democratic theory in particular, investigate epistemic success as a justification for deliberation and voting. Joshua Cohen's classic paper (1986) on a cognitive interpretation of voting lays the foundations. The epistemic dimension of deliberation is emphasized by Nino (1996, ch. 5), Marti (2006), Anderson (2006), Talisse (2009), Estlund (2008), Landemore (2013), Peter (2016), Goodin and Spiekermann (2018, ch. 9) among others. Min and Wong (2018) and Estlund and Landemore (2018) offer detailed reviews. One semi-formal approach has gained particular prominence in the normative literature: the claim that the epistemic success of deliberation and voting depends on the diversity of the group engaging in these practices. This development can be traced to Landemore (2013a, b), who introduced Hong and Page's (2004) and Page's (2007) theorems on the value of diversity in aggregation to democratic theory. The literature emerging from this sheds light on the importance of diversity for epistemic success, but also triggered a debate about the applicability of the model (e.g., Thompson 2014; Kuehn 2017; Benson 2021). While undoubtedly helpful, Hong and Page's model and the conclusions that follow rest on quite specific assumptions. It is not without irony that this literature, despite emphasising the importance of diverse perspectives, is itself lacking diversity as far as the formal models and conceptualizations of deliberation are concerned. Our paper breaks the mould by suggesting a new formal approach to deliberation.

3 A Model of Opinions and Deliberation

This section presents our formal model, in a simple version later generalised in Section 6.

3.1 Opinions and their sources

A group of persons, labelled $1, ..., n$, faces two options, labelled 1 and -1. The group is denoted $N = \{1, ..., n\}$ and has any finite size $n \geq 1$. Following the epistemic 'Condorcetian' paradigm, exactly one option is objectively or intersubjectively *correct*; it is called the *state of the world*, for short the *state*. We represent it by a random variable $\mathbf{x}$ taking the value 1 or -1. In general, we denote random variables in bold letters, their particular values in non-bold letters, and the probability function by 'Pr', all of which refer to some underlying probability space.

Each person forms an opinion about which opinion is correct. There are three possible

Feinberg 2021).

opinions: the opinion that option 1 is correct (labelled 1), the opinion that option -1 is correct (labelled -1), and a neutral or undecided opinion (labelled 0). Opinions are based on 'evidences', in the broadest sense that includes empirical facts, arguments, normative aspects, and other inputs into opinion formation (but we set aside non-evidential 'noise' inputs, captured later in our generalised model). Formally, let S be a finite non-empty set of *sources*, and for each source $s \in S$ let $\mathbf{e}_s$ be a real-valued random variables, the *evidence* from source s. A positive, negative, or zero value of an evidence represents support for option 1, support for option -1, or evidential neutrality, respectively. The strength of this support is represented by the absolute value of the evidence. For instance, if the source s is an argument, then the evidence $\mathbf{e}_s$ measures which opinion it supports, and how strongly.

Each person i accesses some set of sources, her *source set*, represented by a random variable $\mathbf{S}_i$ whose values are subsets of S. In a court jury, a juror's source set might contain a witness report, a legal argument, and a legal text interpreting the law, while another juror's source set might contain the defendant's facial expression when interrogated, and other sources. In the introductory example of Figure 1, the source sets of persons 1, 2, and 3 contain one, two, and one source, respectively.

We can now define several derivative concepts. The *opinion of a person* i is the option supported by i's total evidence:

$$\mathbf{o}_i = \begin{cases} 1 & \text{if } \sum_{s\in\mathbf{S}_i} \mathbf{e}_s > 0 \\ -1 & \text{if } \sum_{s\in\mathbf{S}_i} \mathbf{e}_s < 0 \\ 0 & \text{if } \sum_{s\in\mathbf{S}_i} \mathbf{e}_s = 0. \end{cases}$$

The *majority opinion* is defined by whether opinion 1 or opinion -1 is held by more persons:

$$\mathbf{o}_{maj} = \begin{cases} 1 & \text{if } |\{i : \mathbf{o}_i = 1\}| > |\{i : \mathbf{o}_i = -1\}|, \text{ equivalently } \sum_i \mathbf{o}_i > 0 \\ -1 & \text{if } |\{i : \mathbf{o}_i = 1\}| < |\{i : \mathbf{o}_i = -1\}|, \text{ equivalently } \sum_i \mathbf{o}_i < 0 \\ 0 & \text{if } |\{i : \mathbf{o}_i = 1\}| = |\{i : \mathbf{o}_i = -1\}|, \text{ equivalently } \sum_i \mathbf{o}_i = 0. \end{cases}$$

The *competence* of a person i is the probability of a correct opinion $p_i = Pr(\mathbf{o}_i = \mathbf{x})$. The *majority competence* is the probability of a correct majority opinion $p_{maj} = P(\mathbf{o}_{maj} = \mathbf{x})$.

Diversity is key to successful deliberation.[3] It can be construed as heterogeneity in sources, i.e., dissimilarity between the source sets $\mathbf{S}_i$ (the arguments, the empirical knowledge, etc.) of different persons i. Under minimal diversity, people have identical source sets, so identical opinions. Under maximal diversity, they have pairwise disjoint source sets. A different concept is that of *intrapersonal diversity*. Someone has high intrapersonal diversity if they have a large source set, hence an opinion with a broad basis. As will emerge, deliberation tends to 'internalise' diversity: it lets sources be more widely held, which transforms interpersonal into intrapersonal diversity.

We make three simplifying assumptions (lifted later):

Equiprobable States: the state $\mathbf{x}$ takes both values 1 and -1 with probability $\frac{1}{2}$.

Simple Gaussian Evidences: Given any state $x \in \{\pm 1\}$, the evidences $\mathbf{e}_s$ ($s \in S$) have independent Gaussian distributions with mean x and some variance σ^2 that is the same

[3] For an influential approach to diversity see Hong and Page (2004, 2012) and Page (2007).

across states x and sources s. So, each evidence correlates positively with the state: positive evidence objectively supports state 1, negative evidence objectively supports state -1. This positive correlation is the rationale behind our definition of opinions $\mathbf{o}_i$, according to which each evidence indeed pulls the opinion towards the state of same sign.

Independent Sources: The source-accessing events are independent across people and sources, and jointly independent of the state and the evidences. Formally, for each person $i \in N$ and source $s \in S$, we consider the event that person i accesses source s, '$s \in \mathbf{S}_i$', and we require these source-accessing events to be mutually independent, and jointly independent of the state-evidence combination $(\mathbf{x}, (\mathbf{e}_s)_{s\in S})$.[4]

The probability that a person i accesses a source s will be denoted $p_{s\to i} = Pr(s \in \mathbf{S}_i)$ and called an *access probability.* The access probabilities $(p_{s\to i})_{s\in S, i\in N}$ fully determine the distribution of the source profile $(\mathbf{S}_i)$. How? We use Independent Sources twice. First, the probability that a person i has a source set S_i is the product of the probabilities of accessing any source in S_i and *not* accessing any other source:

$$Pr(S_i) = \left(\prod_{s\in S_i} p_{s\to i}\right)\left(\prod_{s\in S\setminus S_i} \overline{p_{s\to i}}\right) \tag{1}$$

where $\overline{p}$ stands for $1-p$. Second, the probability of an entire source profile (S_i) is the product $\prod_i Pr(S_i)$, with $Pr(S_i)$ given by (1).

To summarise, our formal primitive is a *simple opinion structure*, by which we mean a triple $(\mathbf{x}, (\mathbf{e}_s)_{s\in S}, (\mathbf{S}_i)_{i\in N})$, in short $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$, that contains:

(1) a random variable $\mathbf{x}$, the *state* or *correct option*, taking the value 1 or -1 with equal probability;

(2) a family $(\mathbf{e}_s)$, indexed by some set S of *sources* (non-empty and finite), consisting of real-valued random variables, the *evidences* from these sources, which have state-conditionally independent Gaussian distributions with mean $\mathbf{x}$ and with some fixed variance $\sigma^2 > 0$;

(3) a family $(\mathbf{S}_i)$, indexed by some set $N = \{1, ..., n\}$ of *persons* $(1 \leq n < \infty)$, consisting of random subsets of S, the *source sets* of these persons, with distributions determined by access probabilities $(p_{s\to i})_{s\in S, i\in N}$ via (1), independently across persons and independently of the state and the evidences.

Terminology: The *source profile* is the combination of source sets across persons $(\mathbf{S}_i)_{i\in N}$, in short $(\mathbf{S}_i)$. Person i's *evidence bundle* is the family of evidences from her sources $(\mathbf{e}_s)_{s\in\mathbf{S}_i}$; it is doubly random, through her source set $\mathbf{S}_i$ and the evidences $\mathbf{e}_s$ from her sources s. The *evidence profile* is the combination of evidence bundles across people $((\mathbf{e}_s)_{s\in\mathbf{S}_i})_{i\in N}$, in short $((\mathbf{e}_s)_{s\in\mathbf{S}_i})$.

3.2 The rationality of opinions

Is this opinion model arbitrary from a rationality perspective? The worry is natural, as we presuppose a seemingly naive rationale for forming opinions: adding up one's

[4] For instance, when deputies of Congress form opinions about the effectiveness of a law (the state), whether deputy 1 has listened to (accesses) the verdict of some expert (a source) is independent of which other sources she and other deputies access, and also independent of the law's effectiveness (the state) and the evidence from each source (e.g., the verdict of the expert).

evidences and comparing the sum with zero. In fact, such opinion formation *is* rational in a perfectly classical sense. Why?

Classic rationality requires evaluating opinions (decisions) by expected utility. Given our epistemic setting, let us identify 'utility' with 'correctness level', defined as 1 if the opinion is correct, 0 if it is incorrect, and $\frac{1}{2}$ if it is neutral, i.e., zero. Technically, a person i or her opinion $\mathbf{o}_i$ is *classically rational* if the expected correctness level of $\mathbf{o}_i$ weakly exceeds that of all her possible opinions $\mathbf{o}$. Here, a *possible opinion* of person i is any random variable $\mathbf{o}$ that generates 1, -1 or 0 as a function of i's information $(\mathbf{e}_i)_{i\in\mathbf{S}_i}$; its *correctness level* is 1 if $\mathbf{o} = \mathbf{x}$ (correct opinion), 0 if $\mathbf{o} = -\mathbf{x}$ (false opinion), and $\frac{1}{2}$ if $\mathbf{o} = 0$ (neutral opinion).

Theorem 1 *Under any simple opinion structure* $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$, *the opinion* $\mathbf{o}_i$ *of any person* i *is classically rational.*

Later, non-simple opinion structures will allow for non-rational opinions.

3.3 Deliberation as sharing and absorbing

We construe group deliberation as a process of source transmission between members. To capture this idea, we now define the notion of a *share-absorb process.* Such a process is given by parameters of two types, namely, for each source $s \in S$ and person $i \in N$, a 'sharing probability' $p_{s,i\rightarrow}$ and an 'absorbing probability' $p_{s,i\leftarrow}$, both in $[0, 1]$. The process transforms the initial source profile $(\mathbf{S}_i)$ into a post-deliberation source profile $(\mathbf{S}_i^+)$, in two steps. First, each person i shares each of her initial sources $s \in \mathbf{S}_i$ with an independent probability of $p_{s,i\rightarrow}$. Second, for each source s shared by at least someone, each person i with $s \notin \mathbf{S}_i$ absorbs this source with an independent probability of $p_{s,i\leftarrow}$. The new source set of a person i contains i's initial sources *and* i's absorbed sources: $\mathbf{S}_i^+ = \mathbf{S}_i \cup \{s \in S : i \text{ absorbs } s\}$. The process is defined more formally in Appendix B. Figure 2 gives an illustration, which starts from the introductory example and adds a

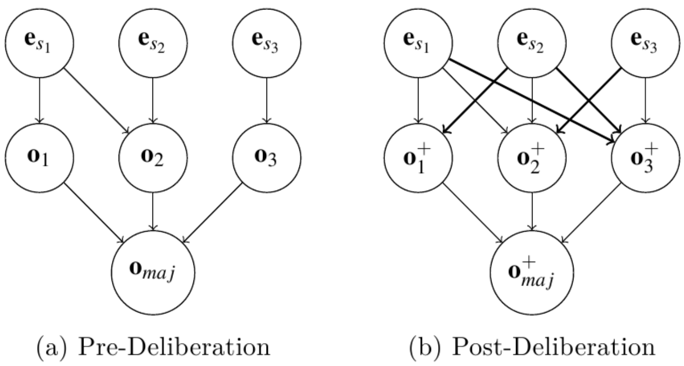

Figure 2: Example of growing access to evidence through deliberation

post-deliberation stage. Thick arrows indicate new sources absorbed during deliberation. The three persons' source sets grow from $S_1 = \{s_1\}$, $S_2 = \{s_1, s_2\}$ and $S_3 = \{s_3\}$ pre-deliberation to $S_1^+ = \{s_1, s_2\}$, $S_2^+ = \{s_1, s_2, s_3\}$ and $S_3^+ = \{s_2, s_3\}$ post-deliberation. To anticipate later sections, this mitigates Failures 1, 2, and 3, because – roughly speaking – sources and their complementarities have become more accessible.

Sharing or absorbing a source can be easy or hard, take seconds or hours, and involve verbal or non-verbal communication. For instance, statistical facts might be easier to share or absorb than complex arguments. So the probabilities $p_{s,i\rightarrow}$ and $p_{s,i\leftarrow}$ can be source-dependent. They can also be person-dependent, partly because some persons are more able or willing than others to share or absorb.

The parameters $p_{s,i\rightarrow}$ and $p_{s,i\leftarrow}$ can have different interpretations: (1) probabilities of *choosing* to share or absorb this source, or (2) probabilities of *succeeding* in the attempt to share or absorb the source, or (3) probabilities of *being capable* at all of sharing or absorbing the source. While we deliberately refrain from modelling micro-foundations (for reasons of parsimony and generality indicated in Section 2), one could interpret the probabilities $(p_{s,i\rightarrow}, p_{s,i\leftarrow})_{s\in S, i\in N}$ as emergent from individual equilibrium strategies in some unmodelled share-receive *game*, whose structure would inevitably depend on whether interpretation (1), (2) or (3) is adopted.[5]

A share-absorb process generates a new opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$ with richer source sets $\mathbf{S}_i^+$. Like the initial opinion structure, the new one induces derivative constructs, namely opinions, competence levels, and (as will soon be seen) imbalance measures capturing failures. They are defined as usual, but based on the new opinion structure; we denote them by the usual symbol with an additional superscript '+'. So, any person i has a new opinion

$$\mathbf{o}_i^+ = \begin{cases} 1 & \text{if } \sum_{s\in\mathbf{S}_i^+} \mathbf{e}_s > 0 \\ -1 & \text{if } \sum_{s\in\mathbf{S}_i^+} \mathbf{e}_s < 0 \\ 0 & \text{if } \sum_{s\in\mathbf{S}_i^+} \mathbf{e}_s = 0 \end{cases}$$

and a new competence p_i^+ $(= Pr(\mathbf{o}_i^+ = \mathbf{x}))$, resulting in a new group opinion $\mathbf{o}_{maj}^+$ and competence p_{maj}^+ $(= Pr(\mathbf{o}_{maj}^+ = \mathbf{x}))$. This machinery will allow us to operationalise our enquiry into the effects of deliberation. For instance, whether deliberation is beneficial overall depends on whether $p_{maj}^+ > p_{maj}$.

Clearly, deliberation creates cross-personal correlations of sources. In result, the new opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$ violates Independent Sources, hence is an opinion structure in the generalised sense of Section 5. Nonetheless our rationality result (Theorem 1) continues to apply, so that post-deliberation opinions remain rational, as shown in Appendix A.

[5] Under interpretation (1), this dynamic game with incomplete information might have the following structure. *Stage 1*: nature randomly draws a state x, evidences $(e_s)_{s\in S}$, personal source sets $(S_i)_{i\in N}$, and some personal character traits, where each person (player) i is informed only of her evidence bundle $(e_s)_{s\in S_i}$ and her character traits. *Stage 2*: simultaneously, each person i chooses which sources in S_i she shares. *Stage 3:* simultaneously, each person i chooses which sources she absorbs among the sources that she did not acquire in Stage 1 and that someone shared in Stage 2. One could include a final *Voting Stage*: simultaneously, everyone casts a vote in $\{1, -1, 0\}$. A player's utility function depends on her character traits, and might reflect that sharing and absorbing are costly and (to capture an epistemic motivation) that successful collective outcomes are valued. If there is a Voting Stage, 'successful' could mean that the voting outcome matches the state. Otherwise, 'successful' could mean that post-deliberation knowledge in the group is high, as measured for instance by the number of persons whose final opinion matches the state. Interpretations (2) or (3) would require further complications, such as somewhere including nature moves determining which sharing or absorbing is successful (for (2)) or at all possible (for (3)).

4 The Wisdom of Crowds Pre- and Post-Deliberation: Two Jury Theorems

The wisdom of crowds is often defended by appealing to jury theorems, but the connection to deliberation has so far remained informal. We now present two jury theorems – one pre-deliberation, one post-deliberation. Compared to classical jury theorems, the message will be revisionary at two levels.

For one, the new jury theorems will draw a less optimistic picture, by setting an objective bound to the wisdom of crowds instead of postulating asymptotically infallible groups. However large, the group cannot beat the 'ideal opinion' – a hypothetical opinion based on total evidence. Yet even the ideal opinion is fallible, because total evidence can lie. Worse, the group can fail to reach the ideal opinion and thus perform 'sub-ideally', because firstly some evidences are accessed by nobody and secondly the accessed evidences are scattered across members and therefore hard to exploit.

Here deliberation steps in, by improving the spread of evidences and thereby helping the group make better use of its evidence and approach the ideal opinion, as our jury theorems suggest. By contrast, classical jury theorems make deliberation appear inessential (as large groups find the truth anyway) or even harmful (by undermining voter independence). This rehabilitation of deliberation is the second revisionary message of our jury theorems.

Our jury theorems operate in the framework of a simple opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ and a share-absorb process, although generalisations would be possible.

4.1 Pre-Deliberation

The *ideal opinion* is the hypothetical opinion based on all sources:

$$\mathbf{o}_{IDEAL} = \begin{cases} 1 & \text{if } \sum_{s\in S} \mathbf{e}_s > 0 \\ -1 & \text{if } \sum_{s\in S} \mathbf{e}_s < 0 \\ 0 & \text{if } \sum_{s\in S} \mathbf{e}_s = 0. \end{cases}$$

The correctness probability of the ideal opinion $p_{IDEAL} = Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$ is the *ideal competence*. As shown in Appendix C, it is the probability that a standard-normal variable takes a value below $\frac{\sqrt{|S|}}{\sigma}$, i.e.,

$$p_{IDEAL} = Pr(\mathbf{o}_{IDEAL} = \mathbf{x}) = F_{N(0,1)}\left(\frac{\sqrt{|S|}}{\sigma}\right), \tag{2}$$

where $F_{N(0,1)}$ is the standard-normal distribution function. The ideal competence is always below 1, reflecting the objective limits of evidence. It is increasing in the number of sources $|S|$ and decreasing in evidence quality σ. For instance, it is $p_{IDEAL} \approx 0.868$ if $|S| = 5$ and $\sigma = 2$.

Since jury theorems vary the group size n, we straightforwardly extend the simple opinion structure $(\mathbf{x}, (\mathbf{e}_s)_{s\in S}, (\mathbf{S}_i)_{i\in N})$ by letting the set of persons N be the infinite set $\{1, 2, ...\}$, called the 'population'. We then talk of a 'simple opinion structure *for an infinite population*'. In such a structure, we can consider groups $\{1, ..., n\} \subseteq N$ of any finite size $n \geq 1$, with a corresponding majority opinion denoted $\mathbf{o}_{maj,n}$ or simply $\mathbf{o}_{maj}$, and a majority competence $Pr(\mathbf{o}_{maj,n} = \mathbf{x})$ denoted $p_{maj,n}$ or simply p_{maj}.

Our first jury theorem says that a finite group performs sub-ideally as long as people are not utterly perfect at accessing sources ('Imperfect Access'), but the group reaches the ideal asymptotically if people are good enough at accessing sources ('Access Competence'). Formally:

Imperfect Access: At least one source $s \in S$ is not surely accessed, i.e., has access probability $p_{i \to s} < 1$ for each person i.

Access Competence: The probability $p_{s \to i}$ that a person $i \in N$ accesses a source $s \in S$ is at least $2^{-1/|S|} + \epsilon$, for some $\epsilon > 0$ independent of i and s.

Pre-Deliberation Jury Theorem: *Given a simple opinion structure* $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ *for an infinite population, the majority competence* $p_{maj,n}$
 (a) *is at most the ideal competence (2), and less than it under Imperfect Access,*
 (b) *converges to the ideal competence (2) as* $n \to \infty$ *under Access Competence.*

Access Competence is very demanding. For example, with $|S| = 5$ sources the access probability $p_{s \to i}$ must exceed $2^{-1/5} \approx 0.87$ for all persons i and sources s. Fortunately, after deliberation a weaker competence assumption suffices. Why?

4.2 Post-Deliberation

Now suppose the group deliberates before voting. So, consider a share-absorb process. To make the process apply to arbitrarily large group sizes n, we assume that its sharing and absorbing probabilities $(p_{s,i\to}, p_{s,i\leftarrow})_{s \in S, i \in N}$ run over the infinite population $N = \{1, 2, ...\}$. We call the so-extended process a share-absorb process *for an infinite population*. For any finite group $\{1, ..., n\} \subseteq N$ (where $n \geq 1$), the extended process induces a standard share-absorb process for this group, defined by the parameters $(p_{s,i\to}, p_{s,i\leftarrow})_{s \in S, i \in \{1,...,n\}}$, i.e., the (sub)family of parameters with persons from $\{1, ..., n\}$. This (sub)process generates a post-deliberation source set $\mathbf{S}^+_{i,n}$ for each group member $i \in \{1, ..., n\}$, and hence a post-deliberation opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}^+_{i,n}))$, with personal opinions $\mathbf{o}^+_{i,n}$, personal competences $p^+_{i,n}$, a group opinion $\mathbf{o}^+_{maj,n}$, and a group competence $p^+_{maj,n}$. All these concepts are defined as usual. The extra index 'n' signals the dependence on the current group size n. Crucially, the same person i can (and will) develop different post-deliberation opinions $\mathbf{o}^+_{i,n}$ depending on the size of the deliberating group: the larger the group, the more sources are shared, hence absorbed. Note that personal post-deliberation competence grows with group size: $p^+_{i,n} \leq p^+_{i,n+1} \leq p^+_{i,n+3} \leq \cdots$ This, however, does not automatically translate into growth of *majority* competence $p^+_{maj,n}$, because the added group members may be less competent. Still, our post-deliberation jury theorem brings good news: majority opinions are asymptotically ideal under a far weaker competence condition than the pre-deliberation competence condition of Access Competence. This weaker competence condition pertains not just to people's ability to access sources initially, but also to their ability to absorb sources during deliberation. We use the label 'acquisition' to refer to both phenomena, initial access and later absorption:

Acquisition Competence: Informally, for all persons i and sources s, the person has a high access probability $p_{s \to i}$ *or* a high absorbing probability $p_{s,i\leftarrow}$ (or both). Formally, for all persons $i \in N$ and sources $s \in S$, the product $(1 - p_{s\to i})(1 - p_{s,i\leftarrow})$ is at most $1 - 2^{-1/|S|} - \epsilon$, for some $\epsilon > 0$ independent of i and s.

Fortunately, if people violate Access Competence because of too low access probabilities, they can still satisfy Acquisition Competence because their absorbing probabilities can make up for their low access probabilities. Deliberation gives them a second chance to acquire sources. Formally:

Proposition 1 *Acquisition Competence is strictly weaker than Access Competence.*

Proof. Given Access Competence, Acquisition Competence holds because, for any $i \in N$ and $s \in S$, $(1 - p_{s\to i})(1 - p_{s,i\leftarrow}) \leq 1 - p_{s\to i} \leq 1 - 2^{-1/|S|} - \epsilon$, where the second '$\leq$' uses Access Competence. Acquisition Competence is *strictly* weaker because under many parameter constellations only Acquisition Competence holds (example: $p_{s\to i} = 0$ and $p_{s,i\leftarrow} = 1$ for all s and i). ■

Our result also uses a minimal condition on participation: new group members do not stop sharing in the limit. This ensures that larger groups have a richer, more diverse deliberation. Technically, recall that $p_{s,i\to}$ represents the *conditional* sharing probability given that the source was accessed in the first place. Our condition pertains instead to the *unconditional* sharing probability, i.e., the probability of accessing and sharing the source, $p_{s\to i} \times p_{s,i\leftarrow}$. We now state our condition, followed by the jury theorem.

Non-Vanishing Participation: For each source $s \in S$, the probability that a person i accesses and shares s, $p_{s\to i} \times p_{s,i\to}$, does not tend to 0 as $i \to \infty$.[6]

Post-Deliberation Jury Theorem: *Given a simple opinion structure* $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ *and a share-absorb process, both for an infinite population, the post-deliberation majority competence* $p^+_{maj,n}$

(a) *is at most the ideal competence (2), and less than it under Imperfect Access,*

(b) *converges to the ideal competence (2) as* $n \to \infty$ *under Acquisition Competence and Non-Vanishing Participation.*

By this theorem, the interplay of deliberation and group increase makes the group opinion asymptotically ideal under interesting assumptions. Going beyond ideal group opinions remains impossible, no matter how much the group deliberates or is increased, because of objectively limited evidence.

An upshot is that deliberation can lead to asymptotically ideal majority opinions even when people are arbitrarily bad at accessing sources (so that Access Competence fails), provided during deliberation they absorb sources well enough and participate at least minimally, i.e., provided they satisfy Acquisition Competence and Non-Vanishing Participation.

4.3 Closing the competence gap: by deliberation or group increase?

Group competence usually falls short of ideal competence (2). The difference $p_{IDEAL} - p_{maj}$ defines the *competence gap.* To reduce it, two instruments are available: deliberation and group increase. How do they complement one another?

[6] This holds for instance if all $p_{s\to i}$ and $p_{s,i\to}$ exceed some fixed level $\epsilon > 0$.

A source $s \in S$ is available to the group if at least someone accesses it. Formally, the *available source set* is the union of personal source sets $\cup_{i=1}^{n}\mathbf{S}_i$. The *relatively ideal opinion* is the opinion based on the available sources, denoted $\mathbf{o}_{ideal,n}$ or simply $\mathbf{o}_{ideal}$, and defined like $\mathbf{o}_{IDEAL}$ but with 'S' replaced by '$\cup_{i=1}^{n}\mathbf{S}_i$'. Its correctness probability $Pr(\mathbf{o}_{ideal} = \mathbf{x})$ is the *relatively ideal competence*, denoted $p_{ideal,n}$ or simply p_{ideal}.

The competence gap $p_{IDEAL} - p_{maj}$ now decomposes into the sum of two gaps:

- *Gap 1* is the gap from the actual to the relatively ideal competence, $p_{ideal} - p_{maj}$, which stems from imperfect use of available sources.
- *Gap 2* is the gap from the relatively ideal to the ideal competence, $p_{IDEAL} - p_{ideal}$, which stems from the unavailability of some sources in S.

Deliberation is an attempt to reduce gap 1. It cannot reduce gap 2 because it does not 'discover' new sources (formally, because the new available set $\cup_{i=1}^{n}\mathbf{S}_i^{+}$ is no larger than the old one $\cup_{i=1}^{n}\mathbf{S}_i$).[7] Gap 2 could instead be reduced by increasing group size. Indeed, $p_{ideal,n}$ converges to p_{IDEAL} as $n \to \infty$, under the minimal assumption that the access probability $p_{s\to i}$ does not converge to 0 as $i \to \infty$. The reason is that, under this assumption of 'non-vanishing access competence', each source is ultimately accessed by *someone* when adding persons.[8] Increasing the group can also reduce gap 1; it even asymptotically closes gap 1 (and 2) under the fortunate conditions of Access Competence, by the Pre-Deliberation Jury Theorem. But normally Access Competence fails, and mere group size increase cannot close gap 1; here deliberation is crucial for closing gap 1. Figure

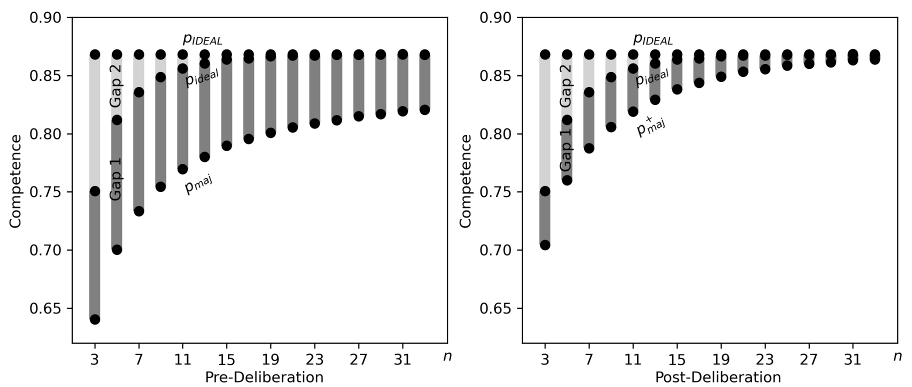

Figure 3: How the competence gaps depend on deliberation and group size

3 shows how deliberation and group size affect the competence gaps, for some typical parameter values chosen such that Access Competence fails while Acquisition Competence holds.[9] Gap 1 shrinks considerably though deliberation, as is seen by comparing the pre- and post-deliberation plots. Gap 2 is deliberation-invariant but shrinks when adding persons, due to increasingly available evidence. The ideal competence p_{IDEAL} (top line) represents a hard upper bound; it is well below 1, underscoring the objective

[7] Under a broader concept of deliberation (formalised in Section 6), deliberation can also discover sources that nobody held initially, and thereby help close also gap 2. This would strengthen the case for deliberation further.

[8] With probability one, the available source set $\cup_{i=1}^{n}\mathbf{S}_i$ converges to the full set S as $n \to \infty$.

[9] Specifically, $|S| = 5$, $\sigma = 2$, $p_{s\to i} = 0.2$, $p_{s,i\to} = 0.5$ and $p_{s,i\leftarrow} = 0.85$. Access Competence fails as $p_{s\to i} < 2^{-1/|S|} \approx 0.871$. Acquisition Competence holds as $(1-p_{s\to i}) \times (1-p_{s,i\leftarrow}) = 0.12 < 1-2^{-1/|S|} \approx 0.129$. The values in Figure 3 were computed using Monte Carlo simulations.

limitation of evidence. Gap 1 persists at all group sizes when not deliberating (left) but disappears asymptotically when deliberating (right). Exactly this was expected from our jury theorems, as Access Competence fails but Acquisition Competence holds.

5 A Typology of Beneficial and Harmful Deliberation

Jury theorems cannot reveal the concrete mechanisms by which deliberation helps or harms. Here, Monte Carlo simulations become useful, if not indispensable. This section uses such simulations to help us understand *how* sharing and absorbing interact to mitigate (or worsen) Failures 1 and 2, and ultimately raise (or lower) group competence. Failure 3 is set aside for now.

To enable failure simulations, we first define numerical proxies of Failures 1 and 2 (Section 5.1). Share-absorb processes turn out to perform well in base-line scenarios (Section 5.2). We will however identify five harmful scenarios (Section 5.3), an analysis of which suggests that deliberation should be 'participatory' and 'even', but not necessarily 'equal' (Section 5.4).

5.1 Imbalance measures as proxies of Failures 1 and 2

Failures 1 and 2 each stem from an imbalance – either between sources (which differ in spread) or between persons (who are evidentially unequal). How can both forms of imbalance be measured?

Spread imbalance. The spread of a source s is the number of source owners $\#\{i : s \in \mathbf{S}_i\}$. The absolute variation of spread between two distinct sources s and s' is $|\#\{i : s \in \mathbf{S}_i\} - \#\{i : s' \in \mathbf{S}_i\}|$. More relevant is the relative or percentage variation. A change of spread from 1 to 3 persons and a change from 101 and 103 persons both represent the same absolute variation (by 2), but the first change represents a much larger *relative* variation. We calculate the relative variation of spread between sources s and s' by dividing the absolute variation by the average spread $\frac{1}{2}\left(\#\{i : s \in \mathbf{S}_i\} + \#\{i : s' \in \mathbf{S}_i\}\right)$. Here and elsewhere, divisions of 0 by 0 are handled by setting $\frac{0}{0} = 0$. We now define the *spread imbalance* as the average relative variation of spread across all pairs of distinct sources:

$$\begin{aligned}\mathbf{SI} &= \frac{1}{|S|\left(|S|-1\right)} \sum_{(s,s') \in S^2 : s \neq s'} \text{'spread imbalance between } s \text{ and } s'\text{'} \\ &= \frac{1}{|S|\left(|S|-1\right)} \sum_{(s,s') \in S^2 : s \neq s'} \frac{|\#\{i : s \in \mathbf{S}_i\} - \#\{i : s' \in \mathbf{S}_i\}|}{\frac{1}{2}\left(\#\{i : s \in \mathbf{S}_i\} + \#\{i : s' \in \mathbf{S}_i\}\right)}.\end{aligned}$$

Here $|S|\left(|S|-1\right)$ is the number of pairs (s, s') of distinct sources.

Interpersonal imbalance. The evidence strength of a person i is her absolute total evidence $\left|\sum_{s \in \mathbf{S}_i} \mathbf{e}_s\right|$. The absolute variation of evidence strength between two distinct persons i and j is $\left|\left|\sum_{s \in \mathbf{S}_i} \mathbf{e}_s\right| - \left|\sum_{s \in \mathbf{S}_j} \mathbf{e}_s\right|\right|$. What matters is, however, the relative or percentage variation of evidence strength between persons i and j, obtained by dividing the absolute variation by the average strength $\frac{1}{2}\left(\left|\sum_{s \in \mathbf{S}_i} \mathbf{e}_s\right| + \left|\sum_{s \in \mathbf{S}_j} \mathbf{e}_s\right|\right)$. We define the *interpersonal imbalance* as the average relative variation of evidence strength across all pairs of distinct persons:

$$\mathbf{II} = \frac{1}{n(n-1)} \sum_{(i,j)\in N^2 : i\neq j} \text{'imbalance between } i \text{ and } j\text{'}$$
$$= \frac{1}{n(n-1)} \sum_{(i,j)\in N^2 : i\neq j} \frac{\left| \left| \sum_{s\in \mathbf{S}_i} \mathbf{e}_s \right| - \left| \sum_{s\in \mathbf{S}_j} \mathbf{e}_s \right| \right|}{\frac{1}{2}\left(\left| \sum_{s\in \mathbf{S}_i} \mathbf{e}_s \right| + \left| \sum_{s\in \mathbf{S}_j} \mathbf{e}_s \right| \right)}.$$

Here, $n(n-1)$ is the number of pairs (i,j) of distinct persons.

Resulting imbalance versus systemic imbalance. **SI** and **II** measure *resulting or ex-post imbalance*, as a consequence of the form taken by the source sets $\mathbf{S}_i$ and evidences $\mathbf{e}_s$. By contrast, *systemic or ex-ante imbalance* is the tendency towards resulting imbalance. We measure systemic imbalance by expected resulting imbalance. Formally, the systemic spread imbalance is $\mathcal{SI} = \mathbb{E}(\mathbf{SI})$ and the systemic interpersonal imbalance is $\mathcal{II} = \mathbb{E}(\mathbf{II})$. We shall often talk of 'imbalance' simpliciter, thereby referring either to systemic imbalance ($\mathcal{SI}$ and $\mathcal{II}$) or to resulting imbalance (**SI** and **II**) – the context will leave no ambiguity.

The imbalance indices as proxies of Failures 1 and 2. Our simulations will use the two imbalance indices – spread imbalance and interpersonal imbalance – as proxies for the extent of Failures 1 and 2, respectively. The rationale is simple: Failure 1 ('overcounting widespread evidence') occurs to the extent that evidences have differently strong spread, which is measured by spread imbalance, and Failure 2 ('neglecting evidential inequality') occurs to the extent that there is evidential inequality, which is measured by interpersonal imbalance.

Just as imbalance can be understood as resulting or systemic imbalance, so Failures 1 and 2 (and 3) can be understood as resulting failures or as systemic failures, i.e., tendencies towards resulting failures. We use **SI** and **II** as proxies for resulting failures, and use $\mathcal{SI}$ and $\mathcal{II}$ as proxies for systemic failures.

5.2 Beneficial deliberation in base-line cases

We now present simulation results. They will show that share-absorb processes can generate diverse but not erratic aggregate phenomena, which can be systematised, explained, and exploited for recommendations. The current subsection treats cases of beneficial deliberation; the next two subsections turn to harmful deliberation and recommendations.

All simulations apply share-absorb processes to simple opinion structures $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$. Previous notation applies. We shall estimate the old and new group competence p_{maj} and p^+_{maj}, and the old and new failure proxies $\mathcal{SI}$, $\mathcal{SI}^+$, $\mathcal{II}$ and $\mathcal{II}^+$ – under various parameter constellations. In principle, one could vary all model parameters: the group size n, the source number $|S|$, the evidence quality σ, the access probabilities $p_{s\to i}$, the sharing probabilities $p_{s,i\to}$, and the absorbing probabilities $p_{s,i\leftarrow}$. While we have explored several parameter constellations privately, we only report results that vary the parameters $p_{s\to i}$, $p_{s,i\to}$ and $p_{s,i\leftarrow}$, while assuming that $n = 9$, $|S| = 5$ and $\sigma = 2$ (one exception will be highlighted). These choices of n, $|S|$ and σ are rich enough for making meaningful comparisons, and limited enough for inspecting results visually and keeping computational costs low. Our private robustness checks for other values of n, $|S|$ and σ suggest that not much is lost by focusing on our particular values of n, $|S|$ and σ.[10] Our estimates are obtained by taking averages over 1,000,000 rounds of Monte Carlo simulation (our Python code is available as Supplementary Material).

We call a share-absorb process:

- *even* if its parameters are source-independent. Intuitively, sources are treated symmetrically.
- *equal* if its parameters are person-independent. Intuitively, everyone takes part equally in deliberation.

The labels 'even' and 'equal' can also be applied to sharing alone, or to absorbing alone, or to access, meaning that the corresponding parameters are source- or person-independent, respectively. Finally, deliberation is:

- *participatory* if every person i shares substantially, in the sense that the average sharing probability $\frac{1}{|S|}\sum_{s\in S} p_{s,i\to}$ exceeds some threshold δ (of for instance 0.5). For even deliberation this condition simply means that everyone's (source-independent) sharing probability exceeds δ. There are stronger and weaker notions of 'participatory', depending on the choice of δ; which one applies is context-dependent.

Figure 4 gives examples of how deliberation performs in the base-line case of even, equal and participatory scenarios. Here $p_{s\to i}$, $p_{s,i\to}$ and $p_{s,i\leftarrow}$ are all independent of s and i, leaving us with just three parameters to vary. In all these scenarios, deliberation raises

	Parameters			**Pre-Deliberation**			**Post-Deliberation**			**change from...**		
#	$p_{s\to i}$	$p_{s,i\to}$	$p_{s,i\leftarrow}$	p_{maj}	$\mathcal{SI}$	$\mathcal{II}$	p^+_{maj}	$\mathcal{SI}^+$	$\mathcal{II}^+$	p_{maj} to p^+_{maj}	$\mathcal{SI}$ to $\mathcal{SI}^+$ in %	$\mathcal{II}$ to $\mathcal{II}^+$ in %
1.1	0.5	0.5	0.5	.839	.394	.842	.859	.300	.602	.020	-24.0	-28.5
1.2	0.8	0.8	0.8	.865	.189	.528	.868	.064	.166	.003	-66.3	-68.5
1.3	0.5	0.8	0.8	.839	.395	.842	.867	.146	.344	.028	-63.0	-59.1
1.4	0.2	0.5	0.5	.755	.847	1.239	.800	.953	.825	.045	12.5	-33.4

Figure 4: Results for even, equal and participatory deliberation.

majority competence and reduces Failure 2. Deliberation occasionally raises Failure 1, as $\mathcal{SI}$ grows in Scenario 1.4, but the effect does not dominate since group competence still grows.

5.3 Five types of harmful deliberation

Outside the even, equal and participatory base-line case, deliberation can become harmful, i.e., lower majority competence. We have identified five elementary types of harmful deliberation; these types and some 'hybrid' types that combine them seem to exhaust the space of harmful share-absorb processes (except for degenerate cases discussed in Section 5.4). Figure 5 gives an example of each elementary type. Remarkably, in all five examples deliberation raises Failure 1, not 2: deliberation harms by raising source imbalance, not evidential inequality. We now discuss the five elementary harmful types. By a 'scenario' we mean a parameter constellation, i.e., a family of access, sharing and

[10] Other values of n, $|S|$ and σ affect the extent and frequency of harmful outcomes, but seem not to add entirely new types of harmful deliberation.

#	Parameters			Pre-Deliberation			Post-Deliberation			change from...		
	$p_{s\to i}$	$p_{s,i\to}$	$p_{s,i\leftarrow}$	p_{maj}	$\mathcal{SI}$	$\mathcal{II}$	p^+_{maj}	$\mathcal{SI}^+$	$\mathcal{II}^+$	p_{maj} to p^+_{maj}	$\mathcal{SI}$ to $\mathcal{SI}^+$ in %	$\mathcal{II}$ to $\mathcal{II}^+$ in %
	Fully private evidence											
2.1	1|0	0.5	0.5	.890	0	.879	.874	.726	.780	-.016	∞	-11.3
	Non-participatory deliberation											
2.2	0.2	0.1	1	.754	.847	1.239	.749	1.013	.822	-.005	19.5	-33.6
	Uneven sharing 1:4											
2.3	0.2	1|0	1	.755	.847	1.239	.740	1.077	.703	-.015	27.1	-43.3
	Uneven absorbing 1:4											
2.4	0.2	1	1|0	.755	.847	1.239	.741	1.077	0.702	-.015	27.2	-43.3
	Unequal sharing 1:8											
2.5	0.2	1|0	1	.756	.847	1.239	.748	1.033	.758	-.007	22.1	-38.8

Figure 5: Examples of the five harmful types of deliberation

absorbing probabilities $(p_{s\to i}, p_{s,i\to}, p_{s,i\leftarrow})_{s\in S, i\in N}$.

Type 1: some private-evidence scenarios. Deliberation harms in some scenarios where most or all members have few or no evidences in common: their source sets have little or no overlap. In Scenario 2.1, each person i accesses only one source s_i, i.e., $\mathbf{S}_i = \{s_i\}$ for sure, where $s_i \neq s_j$ if $i \neq j$. (To model this scenario, we let $|S|$ equal $n = 9$ rather than 5.) Here, the group is already highly competent pre-deliberation, partly because Failure 1 is fully absent and voters are independent. Deliberation unsettles this fine balance, creating Failure 1 and causing evidential overlaps between persons. All this happens despite deliberation being perfectly even and equal. So, 'balanced deliberation' can cause imbalance.

Scenario 2.1 is of special interest because it yields the classic Condorcet jury setting.[11] It is essentially equivalent to Austen-Smith and Banks' (1996) standard jury model, which follows Condorcet but adds the previously implicit informational basis of opinions.[12] Two insights follow. First, since Scenario 2.1's access structure is artificial, classic jury theorems implicitly rely on an implausible opinion structure. Second, this implicit assumption has skewed the debate about the relevance of deliberation for voting: the cards have been stacked against deliberation. Deliberation is far more useful in reality than is being suggested by classic jury settings.

Type 2: some non-participatory scenarios. Deliberation harms in certain scenarios where many voters have low average sharing probability. An example is Scenario 2.2, with a sharing probability of only 0.1. Although Scenario 2.2 is equal and even in both access and deliberation, deliberation surprisingly harms majority competence, driven

[11] It implies Condorcet's controversial assumptions of voter independence and homogeneous competence: voters have independent and identical probabilities above $\frac{1}{2}$ of holding a correct opinion.

[12] Scenario 2.1 and Austen-Smith and Banks' model both let voters access (state-conditionally) independent evidences of homogeneous quality. Scenario 2.1 differs from Austen-Smith and Banks' model in that evidences are Gaussian rather than binary. But this difference is inessential if each voter has a single evidence and hence need not aggregate different evidences.

by rising spread imbalance (Failure 1). By the combination of low sharing and high absorbing, deliberation puts very few evidences on the table; these are then widely absorbed and become overinfluential, letting Failure 1 rise. Figure 6 gives an example of

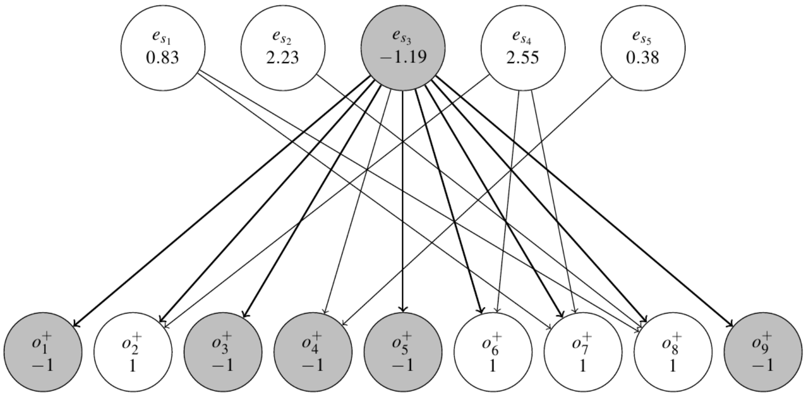

Figure 6: An epistemically harmful outcome of even deliberation

what can happen: only one source s_3 – with misleading evidence – is put on the table, and spreads fully. Thin arrows indicate initial access, thick arrows indicate additional post-deliberation access. The evidence values and post-deliberation opinions are as displayed. The correct option being 1, source s_3 supports the incorrect option, namely -1. By spreading misleading evidence, deliberation turns a correct majority opinion $o_{maj} = 1$ into an incorrect one $o^+_{maj} = -1$. But total available evidence was non-misleading: $e_{s_1} + \cdots + e_{s_5} > 0$. So the *ideal* opinion (cf. Section 4) is correct, and the problem is one of evidence management, not evidence availability.

Types 3 & 4: some uneven-sharing or uneven-absorbing scenarios: Deliberation harms in some scenarios where sources are shared unevenly (see Scenario 2.3) or absorbed unevenly (see Scenario 2.4). The origin of the problem is in plain sight: certain evidences are singled out for wide spread. Deliberation creates a bottleneck where few evidences become dominant, which feeds into Failure 1 and possibly lowers majority competence. The effect is at its worst if very few or just one evidence is put on the table (Scenario 2.3) or picked up (Scenario 2.4).

Type 5: some unequal-sharing scenarios. Deliberation harms in certain scenarios where some members share far more actively than others. For instance, in Scenario 2.5 one member shares everything, the others nothing; the unilaterally shared sources are then widely absorbed. Interestingly, even though this mechanism operates through person- rather than source-dependence, it is yet again Failure 1 (not 2) that rises and causes a fall in collective competence.

Although the five harmful scenarios differ structurally, they share two features. First, members have low access probability on average; otherwise enough evidence is available to prevent the negative deliberation effects identified. Second, as mentioned, deliberation always harms through raising Failure 1, not 2. We did privately identify some rare scenarios where deliberation raised Failure 2, but this never translated into falling majority competence *unless* also Failure 1 rose. So the drawbacks of a deliberative rise in Failure 2 seem to be compensated by the deliberative rise in individual competence.

5.4 Recommendations and discussion

Our analysis yields a clear and robust recommendation for making better group decisions: the group should engage in *participatory and even* deliberation. Such deliberation is characterised by source-independent sharing and absorbing probabilities $p_{s,i\rightarrow} \equiv p_{i\rightarrow}$ and $p_{s,i\leftarrow} \equiv p_{i\leftarrow}$ ('even') and sufficiently high $p_{i\rightarrow}$ ('participatory'). More precisely, our analysis warrants the following general conjecture:

Conjecture: *Deliberation in the form of a participatory and even share-absorb process improves collective competence, under any plausible (simple) opinion structure.*

The Conjecture is warranted because participatory and even deliberation blocks the five harmful types of scenario. Type 2 is blocked because it is non-participatory. Types 3 and 4 are blocked because they are uneven. For types 1 and 5 the point is more subtle. The Scenarios 2.1 and 2.5 exemplifying these harmful types are even; but they are not very participatory as the sharing probability is only $\frac{1}{2}$ in 2.1 and only $\frac{1}{9}$ on average in 2.5. This point generalises: all even scenarios of the harmful type 1 or 5 involve some sharing probabilities significantly below 1 and can be remedied by increasing participation.

The Conjecture excludes 'implausible' initial opinion structures. Indeed, participatory and even deliberation can be non-beneficial for certain highly artificial access parameters. Two such settings stand out. First, deliberation has no effect at all if all sources are certainly accessed by everyone, or more generally if some sources are certainly accessed by everyone and the other sources are never accessed by anyone; in such cases everyone has the same source set, hence learns nothing in deliberation. Second, if some minority of persons certainly access all sources while the other persons never access any sources, then deliberation harms, because the pre-deliberation majority opinion is the minority's opinion (as the other persons abstain), where this opinion is ideal by being based on all sources, whereas the post-deliberation majority opinion can become non-ideal.

Interestingly, while our epistemic approach to democracy suggests participatory and even deliberation, a procedural-fairness approach to democracy might instead suggest participatory *and equal* deliberation, because participation adds legitimacy to outcomes and equality is a central fairness requirement. A combined epistemic-and-procedural approach might therefore recommend participatory, even and equal deliberation.

6 A Generalised Framework

Important real phenomena go beyond the framework used above, with respect to both opinion formation and deliberation. To name just a few limitations, simple opinion structures preclude irrational opinions (cf. Theorem 1) and a proper treatment of Failure 3 (as will be seen). In addition, send-absorb processes preclude deliberative phenomena such as discovery of sources initially outside anyone's access, communication within subgroups or networks, and sharing or absorbing with a bias towards some option. To capture such phenomena and prepare our analysis of Failure 3, we now generalise our model of opinion formation (Section 6.1) and deliberation (Section 6.2).

6.1 General opinion structures

No major departure from simple opinion structures $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ is needed to model opinion formation very generally. It suffices to lift assumptions that were made 'for simplicity'. For instance, by no longer assuming that all $\mathbf{e}_s$ correlate with the state, we can model irrational opinions that are affected by 'noises', i.e., state-independent variables $\mathbf{e}_s$ without objective evidential value. This, for instance, allows modelling verdicts of jurors influenced by the defendant's skin colour, the room temperature, or other noises. In general, $(\mathbf{e}_s)$ will then consist of 'influences', be they evidences or noises. Further, by no longer assuming that all $\mathbf{e}_s$ follow Gaussian distributions, we can model opinion formation as a discrete rather than continuous process, in the simplest case driven by binary influences $\mathbf{e}_s$ taking only the values 1 ('support for 1') and -1 ('support for -1').

Specifically, we lift the three distributional assumptions (Equiprobable States, Simple Gaussian Evidences, and Independent Sources), and moreover we no longer assume that a person i aggregates her personal influences $(\mathbf{e}_s)_{s \in \mathbf{S}_i}$ additively, i.e., we replace the additive expression '$\sum_{s \in \mathbf{S}_i} \mathbf{e}_s$' with a general expression '$g((\mathbf{e}_s)_{s \in \mathbf{S}_i})$'. Here, g is called the 'influence aggregator' and is some function transforming any influence bundle into a real number, the 'total influence'. Simple opinion structures implicitly assume an additive influence aggregator g, given by

$$g((e_s)_{s \in S'}) = \sum_{s \in S'} e_s$$

for any influence bundle $(e_s)_{s \in S'}$ with any source set $S' \subseteq S$.

Formally, a (general) *opinion structure* is thus a quadruple $(\mathbf{x}, (\mathbf{e}_s)_{s \in S}, (\mathbf{S}_i)_{i \in N}, g)$, in short $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i), g)$, that contains:

(1) a random variable $\mathbf{x}$, the *state* or *correct option*, taking the value 1 or -1 with arbitrary non-zero probabilities;

(2) a family $(\mathbf{e}_s)$, indexed by some set S of *sources* (non-empty and finite), consisting of real-valued random variables, the *influences* from these sources, with arbitrary (discrete or continuous) distributions;

(3) a family $(\mathbf{S}_i)$, indexed by some set $N = \{1, ..., n\}$ of *persons* $(1 \leq n < \infty)$, consisting of random subsets of S, the *source sets* of these persons, again with arbitrary distributions;

(4) a function g, the *influence aggregator*, mapping any influence bundle $(e_s)_{s \in S'}$ $(S' \subseteq S)$ to its 'total influence' $g((e_s)_{s \in S'})$ (technically, g is a function from $\cup_{S' \subseteq S} \mathbb{R}^{S'}$ to $\mathbb{R}$).

In the default case of an additive influence aggregator g, we denote the structure by '$(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$', taking additivity for granted. Examples are *simple* opinion structures, which satisfy additivity and the three distributional conditions. An influence $\mathbf{e}_s$ is called a *noise* if it is independent of the state $\mathbf{x}$ (even conditional on the other influences), and an *evidence* otherwise.

As different influences can now be (state-conditionally) dependent, aggregating one's influences additively can in fact be irrational. For instance, *positively* dependent influences, such as similar arguments or evidences from similar sources, are best aggregated subadditively, to avoid double-counting. Fortunately, we allow g to be non-additive – otherwise we would have required irrational responses to correlated evidences. Still g

could be additive, even when this is irrational, i.e., when evidences are correlated. In sum, our model is very flexible and can capture irrational or rational opinions, formed using simple heuristics or sophisticated evidence aggregation.

Our entire earlier machinery carries over to a general opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i), g)$. The *opinion* of a person i is determined by her (now possibly non-additive) aggregate influence:

$$\mathbf{o}_i = \begin{cases} 1 & \text{if } g((\mathbf{e}_s)_{s\in\mathbf{S}_i}) > 0 \\ -1 & \text{if } g((\mathbf{e}_s)_{s\in\mathbf{S}_i}) < 0 \\ 0 & \text{if } g((\mathbf{e}_s)_{s\in\mathbf{S}_i}) = 0. \end{cases}$$

All other derivative concepts – notably the majority opinion $\mathbf{o}_{maj}$, personal competence p_i, majority competence p_{maj}, and spread imbalance **SI** or $\mathcal{SI}$ – keep their original definitions, except that the definition of interpersonal imbalance **II** or $\mathcal{II}$ should be generalised by using g instead of summation to aggregate personal influences. Of course, one should now call $(\mathbf{e}_i)_{i\in\mathbf{S}_i}$ person i's *influence bundle* and call $((\mathbf{e}_i)_{i\in\mathbf{S}_i})$ the *influence profile*, since the earlier labels 'evidence bundle' and 'evidence profile' neglect the possibility of non-evidential influences.

6.2 General deliberation processes

Deliberation may involve phenomena that go beyond a share-absorb process, such as: (1) the discovery of entirely new arguments, aspects or other sources outside anyone's initial awareness (Goodin 2017; more generally: Müller 2018), (2) non-public deliberation, in subgroups or networks, or (3) evidence-sensitive (possibly biased) sharing, where sources are shared only if they provide evidence of certain strength or direction. While these three phenomena can be captured by suitably generalised share-absorb processes, they call for a unified notion of 'deliberation process' that can accommodate these phenomena and others. We now spell out the three processes capturing (1)–(3), before presenting our unified notion of 'deliberation process'. Throughout we presuppose an arbitrary initial opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i), g)$. The three generalised share-absorb processes (and later the unified process) each generate a new source profile $(\mathbf{S}_i^+)$. How?

- *To model (1)*, we add a discovery stage to the share-absorb process, located between sharing and absorbing. Here is an example of the three-stage process: two arguments are shared, then this sparks the discovery of a third argument, and finally the three arguments on the table are selectively absorbed by members. Formally, besides the usual sharing and absorbing probabilities $p_{s,i\rightarrow}$ and $p_{s,i\leftarrow}$, we introduce probabilities $p_{S\rightarrow s}$ of discovering a sources $s \in S$ after a set of sources $T \subseteq S\backslash\{s\}$ was shared. These sharing, absorbing, and discovering probabilities jointly induce a *share-discover-absorb* process.[13] This can, for instance, model the likely discovery of an argument s after such-and-such arguments T are placed on the table: just set the discovery probability $p_{T\rightarrow s}$ high. Standard share-absorb processes emerge if all discovery probabilities are zero.
- *To model (2)*, we must capture sharing to, or absorbing from, a subgroup. For each source s, person i, and subgroup $J \subseteq N\backslash\{i\}$, consider probabilities $p_{s,i\rightarrow J}$

[13] Based on an initial source profile (S_i), *first*, persons i share their sources $s \in S_i$ with independent probabilities of $p_{s,i\rightarrow}$; *second*, letting T be the set of sources shared, the non-shared sources $s \in S\backslash T$ are discovered with independent probabilities of $p_{T\rightarrow s}$; *third*, persons i absorb sources s that they did not own and were shared or discovered with independent probabilities of $p_{s,i\leftarrow}$.

and $p_{s,i\leftarrow J}$ that i shares s to J or absorbs s from J, respectively. These parameters again induce a generalised share-absorb process. This can model not only deliberation in subgroups or networks, but also biased absorbing, where someone absorbs more easily from certain members than from others, perhaps out of prejudice.

- *To model (3)*, the tendency to share or absorb must depend on the influence from the source. For each source s, number $e \in \mathbb{R}$, and person i, consider probabilities $p_{s,e,i\rightarrow}$ and $p_{s,e,i\leftarrow}$ that person i shares (resp. absorbs) source s emitting influence e. These parameters induce a generalised share-absorb process. It can, for instance, model deliberation where only sufficiently influential influences are transmitted: just set $p_{s,e,i\rightarrow}$ and $p_{s,e,i\leftarrow}$ to zero for small $|e|$. It can also model biased sharing, where some members i only share sources whose evidence supports option 1 (so that $p_{s,e,i\rightarrow} = 0$ if $e \leq 0$) while other members i do the opposite (so that $p_{s,e,i\rightarrow} = 0$ if $e \geq 0$). Biased absorbing can be modelled analogously.

What, then, is our unified notion of 'deliberation process' that encompasses all these specific processes and many others? A *deliberation process* is any transformation that stochastically generates a new source profile (S_i^+) based on individual inputs. The input of a person i is anything she has access to, i.e., maximally her influence bundle $(e_s)_{s\in S_i}$. Formally, the process is any (measurable) function D that maps each initial influence profile $((e_s)_{s\in S_i})$, i.e., each value of $((\mathbf{e}_s)_{s\in\mathbf{S}_i})$, to a lottery over source profiles (S_i^+), i.e., profiles of subsets of S. The probability of an (S_i^+) represents how likely (S_i^+) emerges from deliberation, starting from the initial influence profile $((e_s)_{s\in S_i})$. The process generates a new random source profile $(\mathbf{S}_i^+)$, defined as the random source profile whose conditional distribution is $D(((\mathbf{e}_s)_{s\in\mathbf{S}_i}))$ given $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ (hence also given $((\mathbf{e}_s)_{s\in\mathbf{S}_i})$). This generated source profile $(\mathbf{S}_i^+)$ is essentially unique, i.e., its distribution is unique. More generally, the new opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+), g)$ is essentially unique, i.e., its (joint) distribution is unique.[14]

7 A theoretic analysis of Failure 3

Failure 3 arises when some sources emit mutually complementary evidences and this complementarity is underappreciated because few or no voters access all these sources simultaneously. The generalised framework permits modelling complementarity, and hence addressing Failure 3. We shall argue that deliberation mitigates Failure 3 robustly. But first, what *is* complementarity?

Different evidences are 'complementary' if their combination contains different information from the aggregate information of these evidences in isolation. In short: the information of aggregate evidence differs from the aggregate information of evidence. For instance, two arguments are complementary if they imply much information when put together but are each inconclusive in isolation. This then leads to Failure 3 if each person only accesses one of the arguments, so that the complementarity remains unused. Deliberation offers a remedy, by letting persons incorporate both arguments, and hence their complementarity.

More precisely, given an arbitrary opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i), g)$ and particular

[14] That is, if $(\mathbf{S}_i^+)$ and $(\widehat{\mathbf{S}}_i^+)$ are each generated by D, then $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$ and $(\mathbf{x}, (\mathbf{e}_s), (\widehat{\mathbf{S}}_i^+))$ have the same (joint) distribution.

realisations (e_s) of the evidences $(\mathbf{e}_s)$ and (S_i) of the source sets $(\mathbf{S}_i)$,

- Failure 3 *occurs* over sources $S' \subseteq S$ if the evidences $(e_s)_{s\in S'}$ are (i) properly complementary and (ii) dispersed.[15]

Failure 3 may occur over many, few or no set $S' \subseteq S$. Clause (i) is defined soon. Clause (ii) means that each source in S' is accessed by someone (i.e., $S' \subseteq \cup_i S_i$) but someone or everyone fails to access them all (i.e., $S' \nsubseteq S_i$ for some or all i). So, someone or everyone fails to use the complementarity. Worse, fewer persons use the complementarity than use any isolated evidence in $(e_s)_{s\in S'}$, because everyone who uses the complementarity, i.e., accesses all sources in S', necessarily accesses each given source in S'. In short: the complementarity is underappreciated compared to each isolated evidence. Exactly this is Failure 3. The parallel to Failure 1 is striking: While in Failure 1 some evidences have spread less than others, hence is underappreciated, in Failure 3 'complementarity information' has spread less than 'isolated information', hence is underappreciated.

But now assume the group deliberates and achieves a new source profile (S_i^+), following a share-absorb process, or one of its three extensions (cf. Section 6.2), or indeed *any* monotonic deliberation process. 'Monotonic' means that nobody loses sources: $\mathbf{S}_i \subseteq \mathbf{S}_i^+$ for all persons i. Fortunately, the mentioned occurrence of Failure 3 cannot worsen; but it can fall or disappear. Why? Post-deliberation the evidences in $(e_s)_{s\in S'}$ are of course still complementary, but the complementarity is accessed by at least as many (and possibly all) persons. Indeed, every person i who used to access the complementarity, i.e., for whom $S' \subseteq S_i$, still accesses it by monotonicity, and meanwhile new persons may have come to access it.

There are two caveats to our claim that deliberation reduces Failure 3. First, deliberation needs to be monotonic. Second, while deliberation reduces *existing* occurrences of Failure 3, it can create *new* ones if it can discover new sources, as is possible in a share-*discover*-absorb process (Section 6.2). Indeed, under source discovery $\cup_i S_i^+$ grows beyond $\cup_i S_i$ and can thus include new sets S' with complementarity.

The complementarity of an evidence bundle $(e_s)_{s\in S'}$ can be given a precise statistical meaning. How? The central tool is that of *information*. For any evidence bundle $(e_s)_{s\in S'}$, consider a real number $info((e_s)_{s\in S'})$ representing the information in $(e_s)_{s\in S'}$ about the state, i.e., the evidential support for state 1 against state -1. This number could be $g((e_s)_{s\in S'})$, the aggregate evidence according to the opinion structure, or it could be 'statistical information' according to one of the powerful approaches developed in statistics – it will be both if the opinion structure models 'statistically rational' opinion formation. According to the most canonical statistical approach, the information in $(e_s)_{s\in S'}$ is defined by the likelihood-ratio, or equivalently (after changing to a logarithmic scale) by the logarithm of its likelihood-ratio. Formally,

$$info((e_s)_{s\in S'}) = \log \frac{f((e_s)_{s\in S'}|\mathbf{x}=1)}{f((e_s)_{s\in S'}|\mathbf{x}=-1)}, \tag{3}$$

where $f(\cdot|\mathbf{x}=1)$ and $f(\cdot|\mathbf{x}=1)$ denote the joint probability density or mass functions of the relevant bundle conditional on state 1 or -1, respectively.[16] The likelihood-ratio

[15] Slightly abusing language, we call the e_s $(s \in S')$ 'evidences' rather than more generally 'influences', anticipating that clause (i) implies that all e_s $(s \in S')$ are evidences rather than noises.

[16] We assume that the evidences are distributed according to some joint probability density or mass function (conditional on each state), for instance a joint Gaussian density function with some correlation coefficients.

tells us how much more likely state 1 makes the bundle than state -1. If this ratio is above (resp. below, equal to) 1, then its logarithm $info((e_s)_{s\in S'})$ is positive (resp. negative, zero), indicating support for 1 (resp. for -1, for neither). When measuring the information in a single evidence – i.e., when $S' = \{s\}$ – then we simply write '$info(e_s)$' and talk of the 'information 'in e_s'.

We propose to define the *complementarity in* $(e_s)_{s\in S'}$ as the information in this bundle less the information in its parts e_s ($s \in S'$); it represents the 'relational' rather than 'intrinsic' information, contained in the relationship between evidences rather than in the evidences in isolation. Formally, it is a real number $comp((e_s)_{s\in S'})$ that is functionally determined by the combined information $info((e_s)_{s\in S'})$ and each isolated information $info(e_s)$ ($s \in S'$). How exactly should the complementarity depend on the combined and isolated information? This depends on how information was defined. Under the canonical statistical definition of information (3), one should define the complementarity in $(e_s)_{s\in S'}$ by simply subtracting all 'isolated' information from the 'combined' information:

$$comp((e_s)_{s\in S'} = info((e_s)_{s\in S'}) - \sum_{s\in S'} info(e_s);$$

the bundle $(e_s)_{s\in S'}$ is then 'complementarity' simpliciter if $comp((e_s)_{s\in S'} \neq 0$, i.e., if the combined information $info((e_s)_{s\in S'})$ differs from the total isolated information $\sum_{s\in S'} info(e_s)$. Such complementarity is ruled out if evidences are state-conditionally independent.[17] This is why simple opinion structures rule out Failure 3.

Our definition of an occurrence of Failure 3 required the evidences $(e_s)_{s\in S'}$ to be *properly* complementary. What means 'properly'? Informally, the complementarity is improper if it is merely inherited from that of some subbundle(s). If for instance three evidences are complementary merely because the first two are complementary while the third is independent, then the triple is improperly complementary, while the first two are properly complementary. If everyone accesses the first two evidences (but possibly misses the third), and if no other evidences exist (i.e., $|S| = 3$), then there is obviously no complementarity neglect whatsoever, hence no Failure 3. Our definition correctly recognises the absence of any occurrence of Failure 3: there is none over the first two evidences (which are not *dispersed*), none over another pair of evidences (which is not *complementary*), and none over the full triple (which is not *properly* complementary). Formally, a complementary bundle $(e_s)_{s\in S'}$ is *properly* complementary if it cannot be divided into two subbundles $(e_s)_{s\in S^1}$ and $(e_s)_{s\in S^2}$ (with $S^1 \cup S^2 = S'$ and $S^1, S^2 \neq \varnothing$) that are mutually non-complementary, i.e., satisfy $info((e_s)_{s\in S'}) = info((e_s)_{s\in S^1}) + info((e_s)_{s\in S^2})$, or equivalently $comp((e_s)_{s\in S'}) = comp((e_s)_{s\in S^1}) + comp((e_s)_{s\in S^2})$.

8 Concluding Remarks

Knowledge that groups hold in a dispersed fashion is often used poorly because voting is bad at aggregating the knowledge underlying votes. The fundamental tension between respecting voter equality and achieving factually correct decisions is therefore hard to resolve. Giving up the principle of one-person-one-vote is unpalatable, but holding on

[17] Under state-conditional independence, the joint likelihood factorises: $f((e_s)_{s\in S'}|\mathbf{x} = x) = \prod_{s\in S'} f(e_s|\mathbf{x} = x)$ for each state $x \in \{1, -1\}$. Now take the ratio across states and then the logarithm on both sides. This yields $info((e_s)_{s\in S'}) = \sum_{s\in S'} info(e_s)$.

to it is – short of rare symmetries – epistemically suboptimal.

We show that deliberation can mitigate the tension and enable electoral democracies and other groups to make better use of evidence. The effect of deliberation on collective decisions can be studied at two levels: the general level of overall correctness probability of outcomes, or the level of specific failures that threaten collective correctness. At the general level, we have presented the (to our best knowledge) first jury theorems about the effect of deliberation on majority decisions. They suggest that deliberation can increase group competence, though not overcoming the objective limits of available evidence. This message is different from that of orthodox jury theorems, which might suggest that deliberation is unnecessary (because large group perform very well anyway) or even harmful (because voter independence is undermined). At the specific level, we have studied three collective failures: overcounting widespread evidence, neglecting evidential inequality, and neglecting evidential complementarity. A mixture of simulations and theoretic considerations support an overall positive picture: deliberation tends to mitigate these failures, particularly if it is participatory and even, i.e., if everyone contributes substantially and no evidences are privileged over others. A typology of harmful deliberation has been presented.

Taken together, our results provide a robust argument in favor of pre-ballot deliberation on epistemic grounds. Deliberation is not only valuable because democratic citizens owe one another reasons, or because the practice of deliberation is intrinsically valuable, or because deliberation helps structuring voter preferences such as to escape voting paradoxes. It is also valuable because it helps groups make better use of their evidence when voting.

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Appendix

A The Opinions are Rational: Proof

We now prove Theorem 1, in fact generalised to *almost simple* opinion structures $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$. Such opinion structures are defined exactly like simple ones except that Independent Sources is weakened to *Almost Independent Sources*, the condition that the source-access events '$s \in \mathbf{S}_i$' (where $s \in S$ and $i \in N$) are independent across sources s and jointly independent of the state and the evidences, i.e., of $(\mathbf{x}, (\mathbf{e}_s))$. This condition weakens Independent Sources by no longer requiring independence across *persons* of the source-access events. The generalisation ensures that the theorem also captures post-deliberation opinions. Indeed, a share-absorb process transforms a *simple* opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ into an *almost simple* one $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$ – 'almost' because of interpersonal source dependencies.

We begin by proving an astonishing fact about Gaussian evidences:

Lemma 1 *If an opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ satisfies Simple Gaussian Evidences (e.g., is almost simple), then each evidence is proportional to its own log-likelihood-ratio, more precisely*

$$\mathbf{e}_s = \frac{\sigma^2}{2} \log \frac{f(\mathbf{e}_s|1)}{f(\mathbf{e}_s|-1)} \text{ for each } s \in S,$$

where $f(\cdot|x)$ denotes the Gaussian density ('likelihood') function of each evidence $\mathbf{e}_s$ ($s \in S$) given state x ($\in \{\pm 1\}$).

Proof. Let $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ satisfy Simple Gaussian Evidences. Let $s \in S$. Conditional on a state x ($\in \{\pm 1\}$), $\mathbf{e}_s$ is normally distributed with mean x and variance σ^2, hence has a Gaussian density function given by

$$f(e|x) = \frac{1}{\sigma(2\pi)^{1/2}} e^{-\frac{1}{2}\left(\frac{e-x}{\sigma}\right)^2} \text{ for all } e \in \mathbb{R}.$$

We have $\mathbf{e}_s = \frac{\sigma^2}{2} \log \frac{f(\mathbf{e}_s|1)}{f(\mathbf{e}_s|-1)}$ because, for all values $e \in \mathbb{R}$ of $\mathbf{e}_s$,

$$\begin{aligned}\log \frac{f(e|1)}{f(e|-1)} &= \log \frac{\exp\left(-\frac{1}{2}\left(\frac{e-1}{\sigma}\right)^2\right)}{\exp\left(-\frac{1}{2}\left(\frac{e+1}{\sigma}\right)^2\right)} = \log\left[\exp\left(\frac{1}{2}\left(\frac{e+1}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{e-1}{\sigma}\right)^2\right)\right] \\ &= \frac{1}{2}\left(\frac{e+1}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{e-1}{\sigma}\right)^2 = \frac{1}{2\sigma^2}\left[(e+1)^2 - (e-1)^2\right] \\ &= \frac{1}{2\sigma^2}\left[\left(e^2+2e+1\right) - \left(e^2-2e+1\right)\right] = \frac{2}{\sigma^2}e. \blacksquare\end{aligned}$$

Proof of Theorem 1 generalised to almost simple opinion structures. Fix an almost simple opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$, a person i, and a possible opinion of i, i.e., a random variable $\mathbf{o}$ generating values in $\{1, 0, -1\}$ based on i's evidence bundle $(\mathbf{e}_s)_{s \in \mathbf{S}_i}$. We must show that $\mathbb{E}(u(\mathbf{o}_i, \mathbf{x})) \geq \mathbb{E}(u(\mathbf{o}, \mathbf{x}))$, where the utility of any opinion-state pair (o, x) in $\{1, 0, -1\} \times \{1, -1\}$ is the correctness level, given by

$$u(o,x) = \begin{cases} 1 & \text{if } o = x \text{ (correct opinion)} \\ 0 & \text{if } o = -x \text{ (false opinion)} \\ \frac{1}{2} & \text{if } o = 0 \text{ (neutral opinion).} \end{cases}$$

We prove this by showing that $\mathbb{E}(u(\mathbf{o}_i, \mathbf{x})|(\mathbf{e}_s)_{s \in \mathbf{S}_i}) \geq \mathbb{E}(u(\mathbf{o}, \mathbf{x})|(\mathbf{e}_s)_{s \in \mathbf{S}_i})$.[18] So, we fix any value $(e_s)_{s \in S_i}$ of $(\mathbf{e}_s)_{s \in \mathbf{S}_i}$ and, writing o_i (resp. o) for the value of $\mathbf{o}_i$ (resp. $\mathbf{o}$) under $(e_s)_{s \in S_i}$, we prove that

$$\mathbb{E}(u(o_i, \mathbf{x})|(e_s)_{s \in S_i}) \geq \mathbb{E}(u(o, \mathbf{x})|(e_s)_{s \in S_i}). \tag{4}$$

To be able to prove (4), we first establish that

$$Pr(\mathbf{x} = 1|(e_s)_{s \in S_i}) > (<, =)\ \frac{1}{2} \Leftrightarrow \sum_{s \in S_i} e_s > (<, =)\ 0. \tag{5}$$

We only prove the equivalence for '>', as those for '<' and '=' are analogous. Writing $f(\cdot|x)$ for the Gaussian density function of each evidence given state x ($\in \{\pm 1\}$), we have

$$\begin{aligned} Pr(\mathbf{x} = 1|(e_s)_{s \in S_i}) > \tfrac{1}{2} \quad &\Leftrightarrow \quad \tfrac{f((e_s)_{s \in S_i}|1)}{f((e_s)_{s \in S_i}|-1)} > 1 \\ &\Leftrightarrow \quad \textstyle\prod_{s \in S_i} \frac{f(e_s|1)}{f(e_s|-1)} > 1 \\ &\Leftrightarrow \quad \textstyle\sum_{s \in S_i} \log \frac{f(e_s|1)}{f(e_s|-1)} > 0 \\ &\Leftrightarrow \quad \textstyle\sum_{s \in S_i} e_s > 0. \end{aligned}$$

Here, the first equivalence follows easily from Bayes' rule, using that $Pr(\mathbf{x} = 1) = Pr(\mathbf{x} = -1)$ and also that i's source set is independent of the state and the evidences.[19]

[18] Strictly speaking, we must show that this inequality holds *for some versions* of the conditional expectations on both sides. This qualification is necessary because conditional expectations are random variables that are not unique, but still 'essentially unique' in that any two versions of a conditional expectation coincide outside a zero-probability event.

[19] How is this independence condition used here? Informally, it ensures that the identity of the source set S_i is of no extra information, i.e., that only the evidences from those sources carry information. This explains why the numerator and denominator of the likelihood-ratio each features the joint (state-conditional) likelihood of the evidences only, not of the evidences *and* the source set S_i. To be slightly more explicit, conditionalising on the evidence bundle $(e_s)_{s \in S_i}$ is equivalent to conditionalising first on the source set S_i and then on the evidences from these sources; which however reduces to conditionalising only on the evidences, by the independence condition.

The second equivalence holds by state-conditional independence of the evidences. The third equivalence holds by applying the logarithm on both sides of the previous inequality. The fourth equivalence holds by Lemma 1.

We can now prove (4), by proceeding case by case.

Case 1: $o_i = 1$. Then $\mathbb{E}(u(o_i, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) > \frac{1}{2}$, where the inequality holds by (5) as $\sum_{s\in S_i} e_s > 0$.

Subcase 1.1: $o = 1$. Then (4) holds (with '=') because $o_i = o$.

Subcase 1.2: $o = -1$. Then (4) holds (with '>') because $\mathbb{E}(u(o, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = -1|(e_s)_{s\in S_i}) = 1 - Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) < \frac{1}{2}$, where the last inequality holds as $Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) > \frac{1}{2}$.

Subcase 1.3: $o = 0$. Then (4) holds (with '>') because $\mathbb{E}(u(o, \mathbf{x})|(e_s)_{s\in S_i}) = \frac{1}{2}$.

Case 2: $o_i = -1$. Then $\mathbb{E}(u(o_i, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = -1|(e_s)_{s\in S_i}) = 1 - Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) > \frac{1}{2}$, where the inequality holds because $Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) < \frac{1}{2}$ by (5) as $\sum_{s\in S_i} e_s < 0$. An argument similar to that in Case 1 then implies (4).

Case 3: $o_i = 0$. Then $\mathbb{E}(u(o_i, \mathbf{x})|(e_s)_{s\in S_i}) = \frac{1}{2}$. Further, $Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = -1|(e_s)_{s\in S_i}) = \frac{1}{2}$, by (5) as $\sum_{s\in S_i} e_s = 0$.

Subcase 3.1: $o = 0$. Then (4) holds (with '=') because $o_i = o$.

Subcase 3.2: $o = 1$. Then (4) holds (with '=') because $\mathbb{E}(u(o, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) = \frac{1}{2}$.

Subcase 3.3: $o = -1$. Then (4) holds (with '=') because $\mathbb{E}(u(o, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = -1|(e_s)_{s\in S_i}) = \frac{1}{2}$. ■

B The Analytics of Share-Absorb Processes

The definition of share-absorb processes has been stated informally. The formalisation is obvious. In short, given a simple opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ (we could have used a general opinion structure), the share-absorb process with parameters $(p_{s,i\rightarrow}, p_{s,i\leftarrow})_{s\in S, i\in N}$ assumes that there exist events 'i shares s' and 'i absorbs s' for any person $i \in N$ and source $s \in S$; that the new source set of any person i is $\mathbf{S}_i^+ = \mathbf{S}_i \cup \{s \in S : \text{'}i \text{ absorbs } s\text{'}\}$, the set of initially accessed or later absorbed sources; that, for any person i and source s, the probability of 'i shares s' given any initial source profile (S_j) is $p_{s,i\rightarrow}$ if $s \in S_i$ and 0 otherwise$_i$; that, for any person i and source s, the probability of 'i absorbs s' given any initial source profile (S_j) and any sharing profile is $p_{s,i\leftarrow}$ if [$s \notin S_i$ and someone shares s in the sharing profile] and 0 otherwise (where a 'sharing profile' is a combination of truth values of the sharing events across persons and sources); and, finally, that the access, sharing, and absorbing events are jointly independent of the state and the evidences.

How is the new source profile $(\mathbf{S}_i^+)$ distributed? And how is it distributed conditional on the initial evidence profile? We now answer both questions. The first answer completes the description of the new opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$, as we already know how $(\mathbf{x}, (\mathbf{e}_s))$ is distributed and that $(\mathbf{S}_i^+)$ is independent of $(\mathbf{x}, (\mathbf{e}_s))$. The second answer implies an alternative (and equivalent) definition of the share-absorb process as a deliberation process in the general sense of Section 6, i.e., a mapping D from initial evidence profiles to lotteries over new source profiles.

Fix a simple[20] opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ and a share-absorb process with sharing and absorbing probabilities $(p_{s,i\rightarrow}, p_{s,i\leftarrow})_{s\in S, i\in N}$, generating a new source profile

[20]Simplicity could be weakened considerably, to Independent Sources.

$(\mathbf{S}_i^+)$. The probability of any new source profile (S_i^+) (any value of $(\mathbf{S}_i^+)$) is

$$Pr((S_i^+)) = \prod_{s\in S} \pi_s \tag{6}$$

where, for each source $s \in S$, π_s is the probability that the new set of owners of s is $I_s = \{i : s \in S_i^+\}$, and equals

$$\pi_s = \left(\prod_{i\in I_s} p_{s\to i}\right)\left(\prod_{i\in\overline{I_s}} \overline{p_{s\to i}}\right)\left(\prod_{i\in I_s} \overline{p_{s,i\to}}\right) + \left(\prod_{i\in\overline{I_s}} \overline{p_{s,i\leftarrow}}\right)\sum_{\varnothing\neq I\subseteq I_s}\left(\prod_{i\in I} p_{s\to i}\right)\left(\prod_{i\in\overline{I}} \overline{p_{s\to i}}\right)\left(\overline{\prod_{i\in I} \overline{p_{s,i\to}}}\right)\left(\prod_{i\in I_s\setminus I} p_{s,i\leftarrow}\right).$$

Further, the conditional probability of any new source profile (S_i^+) given any initial evidence profile $((e_s)_{s\in S_i})$, or given just (S_i), is

$$Pr((S_i^+)|((e_s)_{s\in S_i}) = Pr((S_i^+)|(S_i)) = \prod_{s\in S} \gamma_s \tag{7}$$

where, for each source $s \in S$, γ_s is the probability that the new set of owners of source s is $I_s = \{i : s \in S_i^+\}$ given that the initial one is $J_s = \{i : s \in S_i\}$, which equals

$$\gamma_s = \begin{cases} \left(\overline{\prod_{i\in J_s} \overline{p_{s,i\to}}}\right)\left(\prod_{i\in I_s\setminus J_s} p_{s,i\leftarrow}\right)\left(\prod_{i\in\overline{I_s}} \overline{p_{s,i\leftarrow}}\right) & \text{if } J_s \subsetneq I_s \\ \left(\overline{\prod_{i\in J_s} \overline{p_{s,i\to}}}\right)\left(\prod_{i\in\overline{J_s}} \overline{p_{s,i\leftarrow}}\right) + \prod_{i\in J_s} \overline{p_{s,i\to}} & \text{if } J_s = I_s \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Proof of (6). For each $s \in S$, define two random subgroups, the old set of owners $\mathbf{J}_s = \{i : s \in \mathbf{S}_i\}$ and the new one $\mathbf{I}_s = \{i : s \in \mathbf{S}_i^+\}$. Fix any (S_i^+), and define each I_s $(s \in S)$ as above. Note that $(\mathbf{S}_i^+)$ takes the value (S_i^+) if and only if $(\mathbf{I}_s)$ $(= (\mathbf{I}_s)_{s\in S})$ takes the value (I_s) $(= (I_s)_{s\in S})$. Hence, $Pr((S_i^+)) = Pr((I_s))$. The sets $\mathbf{I}_s$ are independent across sources s. So, $Pr((I_s)) = \prod_{s\in S} Pr(I_s)$, and thus

$$Pr((S_i^+)) = \prod_{s\in S} Pr(I_s).$$

Now fix a source s. We calculate $Pr(I_s)$ $(= \pi_s)$. We do this under the assumption that all parameters $p_{s\to i}$ and $p_{s,i\to}$ are strictly between 0 and 1. This is sufficient since the formula generalises to extreme parameter values by a continuity argument.

First assume $I_s = \varnothing$. Note that $\mathbf{I}_s$ takes the value $\varnothing$ if and only if $\mathbf{J}_s$ takes the value $\varnothing$. The probability of the latter is $\prod_i \overline{p_{s\to i}}$. So, $Pr(I_s) = \prod_i \overline{p_{s\to i}}$. This is what had to be proved, since the claimed expression for $Pr(I_s)$ $(= \pi_s)$ indeed reduces to $\prod_i \overline{p_{s\to i}}$ if $I_s = \varnothing$.

From now on assume $I_s \neq \varnothing$. Then it is certain that $\mathbf{J}_s$ takes a value J_s that satisfies $\varnothing \neq J_s \subseteq I_s$, and each such value J_s has non-zero probability (as each $p_{s\to i}$ is strictly between 0 and 1), so can be conditionalised on. By implication,

$$Pr(I_s) = \sum_{\varnothing\subsetneq J_s\subseteq I_s} Pr(J_s)Pr(I_s|J_s). \tag{9}$$

In this expression, the term $Pr(J_s)$ can be written as

$$Pr(J_s) = \left(\prod_{i \in J_s} p_{s \to i}\right)\left(\prod_{i \in \overline{J_s}} \overline{p_{s \to i}}\right).$$

We now calculate $Pr(I_s|J_s)$. Denote by $!_s$ the event that at least someone shares s. Given the (non-empty) event J_s, each of $!_s$ and $\overline{!_s}$ has non-zero probability (as each $p_{s,i\to}$ is strictly between 0 and 1), so can be conditionalised on. Hence, $Pr(I_s|J_s)$ is writable as $Pr(!_s|J_s)Pr(I_s|!_s, J_s) + Pr(\overline{!_s}|J_s)Pr(I_s|\overline{!_s}, J_s)$, where $Pr(I_s|\overline{!_s}, J_s)$ is 0 if $J_s \neq I_s$ and 1 if $J_s = I_s$. So,

$$Pr(I_s|J_s) = \begin{cases} Pr(!_s|J_s)Pr(I_s|!_s, J_s) & \text{if } J_s \neq I_s \\ Pr(!_s|J_s)Pr(I_s|!_s, J_s) + Pr(\overline{!_s}|J_s) & \text{if } J_s = I_s. \end{cases}$$

Note that if $J_s = I_s$ then

$$Pr(\overline{I_s}|J_s) = \prod_{i \in I_s} \overline{p_{s,i\to}}.$$

Upon inserting the derived expressions into (9) and rearranging,

$$\begin{aligned} Pr(I_s) = &\left(\prod_{i \in I_s} p_{s \to i}\right)\left(\prod_{i \in \overline{I_s}} \overline{p_{s \to i}}\right)\left(\prod_{i \in I_s} \overline{p_{s,i\to}}\right) \\ &+ \sum_{\varnothing \subsetneq J_s \subseteq I_s} \left(\prod_{i \in J_s} p_{s \to i}\right)\left(\prod_{i \in \overline{J_s}} \overline{p_{s \to i}}\right) Pr(!_s|J_s)Pr(!_s|!_s, J_s). \end{aligned}$$

In this,

$$Pr(!_s|J_s)Pr(I_s|!_s, J_s) = \left(\overline{\prod_{i \in J_s} \overline{p_{s,i\to}}}\right)\left(\prod_{i \in I_s \backslash J_s} p_{s,i\leftarrow}\right)\left(\prod_{i \in \overline{I_s}} \overline{p_{s,i\leftarrow}}\right).$$

So, after rearranging and relabelling the index 'J_s' into 'I',

$$\begin{aligned} Pr(I_s) = &\left(\prod_{i \in I_s} p_{s \to i}\right)\left(\prod_{i \in \overline{I_s}} \overline{p_{s \to i}}\right)\left(\prod_{i \in I_s} \overline{p_{s,i\to}}\right) \\ &+ \left(\prod_{i \in \overline{I_s}} \overline{p_{s,i\leftarrow}}\right) \sum_{\varnothing \neq I \subseteq I_s} \left(\prod_{i \in I} p_{s \to i}\right)\left(\prod_{i \in \overline{I}} \overline{p_{s \to i}}\right)\left(\overline{\prod_{i \in I} \overline{p_{s,i\to}}}\right)\left(\prod_{i \in I_s \backslash I} p_{s,i\leftarrow}\right). \blacksquare \end{aligned}$$

Proof of (7). Fix any initial evidence profile $((e_s)_{s \in S_i})$ and new source profile (S_i^+). Notation is as above. By definition of share-absorb processes, $Pr((S_i^+)|((e_s)_{s \in S_i})) = Pr((S_i^+)|(S_i))$. I_s and J_s are instances of the random variables $\mathbf{I}_s$ and $\mathbf{J}_s$ defined in the proof of (6). Since the events $(\mathbf{S}_i^+) = (S_i^+)$ and $(\mathbf{I}_s) = (I_s)$ are equivalent, and the events $(\mathbf{S}_i) = (S_i)$ and $(\mathbf{J}_{s'}) = (J_{s'})$ are also equivalent,

$$Pr((S_i^+)|(S_i)) = Pr((I_s)|(J_{s'})) = \prod_{s \in S} Pr(I_s|(J_{s'})) = \prod_{s \in S} \underbrace{Pr(I_s|J_s)}_{\gamma_s},$$

where the second and third equalities hold by construction of share-absorb processes.

Now fix a source $s \in S$. It remains to prove that $Pr(I_s|J_s)$ $(= \gamma_s)$ is given by (8). We do this under the assumption that each $p_{s,i\rightarrow}$ is strictly between 0 and 1. (The generalisation to extreme parameters follows by continuity.)

If $J_s = I_s = \varnothing$, then $Pr(I_s|J_s) = 1$, because if no one initially owns s, then certainly no one shares or absorbs s.

If $J_s = \varnothing$ and $I_s \neq \varnothing$, then $Pr(I_s|J_s) = 0$, because a source that no one owns is never shared, hence never acquired.

If $J_s \not\subseteq I_s$, i.e., if J_s is not a subset of I_s, then $Pr(I_s|J_s) = 0$, because during deliberation no one loses any initially held sources.

Now assume the remaining case that $\varnothing \neq J_s \subseteq I_s$. As in the proof of (6), denote by $!_s$ the event that at least someone shares s. Given (S_i), each of $!_s$ and $\overline{!_s}$ has non-zero probability (because the parameters $p_{s,i\rightarrow}$ are neither 0 nor 1, and in case of $!_s$ also because $J_s \neq \varnothing$). So we can conditionalise on $!_s$ and on $\overline{!_s}$, and write

$$Pr(I_s|J_s) = Pr(!_s|J_s)Pr(I_s|!_s, J_s) + Pr(\overline{!_s}|J_s)Pr(I_s|\overline{!_s}, J_s).$$

Hence, as $Pr(I_s|\overline{!_s}, J_s)$ is 0 if $J_s \neq I_s$ and 1 if $J_s = I_s$,

$$Pr(I_s|J_s) = \begin{cases} Pr(!_s|J_s)Pr(I_s|!_s, J_s) & \text{if } \varnothing \neq J_s \subsetneq I_s \\ Pr(!_s|J_s)Pr(I_s|!_s, J_s) + Pr(\overline{!_s}|J_s) & \text{if } \varnothing \neq J_s = I_s \end{cases}$$

In this,

$$\begin{aligned} Pr(\overline{!_s}|J_s) &= \prod_{i \in J_s} \overline{p_{s,i\rightarrow}} \\ Pr(!_s|J_s) &= \overline{\prod_{i \in J_s} \overline{p_{s,i\rightarrow}}} \\ Pr(I_s|!_s, J_s) &= \left(\prod_{i \in I_s \setminus J_s} p_{s,i\leftarrow} \right) \left(\prod_{i \in \overline{I_s}} \overline{p_{s,i\leftarrow}} \right). \end{aligned}$$

Here, $Pr(I_s|!_s, J_s)$ reduces to $\prod_{i \in \overline{J_s}} \overline{p_{s,i\leftarrow}}$ if $J_s = I_s$. In sum, we have shown that

$$Pr(I_s|J_s) = \begin{cases} 1 & \text{if } J_s = I_s = \varnothing \\ 0 & \text{if } \varnothing = J_s \subsetneq I_s \\ 0 & \text{if } J_s \not\subseteq I_s \\ \left(\overline{\prod_{i \in J_s} \overline{p_{s,i\rightarrow}}}\right) \left(\prod_{i \in I_s \setminus J_s} p_{s,i\leftarrow}\right) \left(\prod_{i \in \overline{I_s}} \overline{p_{s,i\leftarrow}}\right) & \text{if } \varnothing \neq J_s \subsetneq I_s \\ \left(\overline{\prod_{i \in J_s} \overline{p_{s,i\rightarrow}}}\right) \left(\prod_{i \in \overline{J_s}} \overline{p_{s,i\leftarrow}}\right) + \prod_{i \in J_s} \overline{p_{s,i\rightarrow}} & \text{if } \varnothing \neq J_s = I_s \end{cases}$$

Of these six cases, the first can be subsumed under the last, as the formula in the last reduces to 1 if $J_s = I_s = \varnothing$; and the second can be subsumed under the fourth, as the formula in the fourth reduces to 0 if $\varnothing = J_s$. This yields formula (8). ■

C The Jury Theorems: Proofs

Proof of equation (2). Under the given assumptions, $\mathbf{o}_{IDEAL} = \mathbf{x}$ holds if and only if total evidence $\sum_{s \in S} \mathbf{e}_s$ has the same sign as $\mathbf{x}$. The probability of this event equals the conditional probability that $\sum_{s \in S} \mathbf{e}_s > 0$ given $\mathbf{x} = 1$, by Simple Gaussian Evidences.

Given $\mathbf{x} = 1$, $\sum_{s \in S} \mathbf{e}_s$ is the sum of $|S|$ independent Gaussian variables of mean 1 and variance σ^2, hence is itself a Gaussian variable, with mean $|S|$ and variance $|S|\,\sigma^2$. The probability that such a variable is positive equals the probability that a standard-Gaussian variable is below $\frac{\sqrt{|S|}}{\sigma}$, by a simple linear transformation. ■

Proof of the Pre-Deliberation Jury Theorem. Assume a simple opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ for an infinite population $N = \{1, 2, ...\}$. Notation is as usual.

(a) To prove the non-asymptotic claim, we fix a group size n and write $\mathbf{o}_{maj}$ for $\mathbf{o}_{maj,n}$. We first show that $p_{maj} \leq p_{IDEAL}$, i.e., that $Pr(\mathbf{o}_{maj} = \mathbf{x}) \leq Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$. We begin by proving a general claim:

Claim: For every discrete random variable $\mathbf{z}$ that is independent of the state-evidence combination $(\mathbf{x}, (\mathbf{e}_s))$ (e.g., for $\mathbf{z} = (\mathbf{S}_i)$),

$$Pr(\mathbf{o}_{IDEAL} = \mathbf{x} | (\mathbf{e}_s), \mathbf{z}) > \frac{1}{2}$$

except in a zero-probability event (i.e., except if the combination $((\mathbf{e}_s), \mathbf{z})$ falls into a set into which it falls with zero probability).

To show the claim, note first that such a variable $\mathbf{z}$ is independent of the event $\mathbf{o}_{IDEAL} = \mathbf{x}$ conditional on $(\mathbf{e}_s)$, because $\mathbf{o}_{IDEAL}$ is a function of $(\mathbf{e}_s)$. So, $Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z})$ can be replaced by $Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s))$, which, by construction of the ideal opinion $\mathbf{o}_{IDEAL}$, indeed exceeds $\frac{1}{2}$, except in the zero-probability event that $\sum_s \mathbf{e}_s = 0$ (i.e., except if $\mathbf{o}_{IDEAL}$ is zero, hence certainly distinct from $\mathbf{x}$). Q.e.d.

Now choose $\mathbf{z} = (\mathbf{S}_i)$. Then

$$Pr(\mathbf{o}_{maj} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) = \begin{cases} Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) & \text{if } \mathbf{o}_{maj} = \mathbf{o}_{IDEAL} \\ 1 - Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) & \text{if } \mathbf{o}_{maj} = -\mathbf{o}_{IDEAL} \\ 0 & \text{if } \mathbf{o}_{maj} = 0. \end{cases} \tag{10}$$

Here, 'if $\mathbf{o}_{maj} = \mathbf{o}_{IDEAL}$' of course means 'if $((\mathbf{e}_s), \mathbf{z})$ takes a value such that $\mathbf{o}_{maj} = \mathbf{o}_{IDEAL}$', which is indeed a well-defined condition because the value of $((\mathbf{e}_s), \mathbf{z})$ determines the values of $\mathbf{o}_{maj}$ and $\mathbf{o}_{IDEAL}$, hence determines whether $\mathbf{o}_{maj} = \mathbf{o}_{IDEAL}$. The meanings of 'if $\mathbf{o}_{maj} = -\mathbf{o}_{IDEAL}$' and 'if $\mathbf{o}_{maj} = 0$' are analogous.

The 'Claim' and (10) jointly imply that, still for $\mathbf{z} = (\mathbf{S}_i)$,

$$Pr(\mathbf{o}_{maj} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) \leq Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) \tag{11}$$

with probability one. By taking expectations on both sides (thereby averaging out $(\mathbf{e}_s)$ and $\mathbf{z}$), we obtain $Pr(\mathbf{o}_{maj} = \mathbf{x}) \leq Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$, i.e., $p_{maj} \leq p_{IDEAL}$.

Finally, assume Imperfect Access. Then with non-zero probability the variable $\mathbf{z} = (\mathbf{S}_i)$ takes a value such that some source is not accessed by anyone, hence not accessed by a majority. This easily implies that with non-zero probability the second or third case in (10) applies. So, in (11) the '$\leq$' is a '$<$' with non-zero probability. Hence, taking the expectation on both sides of (11) now yields $Pr(\mathbf{o}_{maj} = \mathbf{x}) < Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$, i.e., $p_{maj} < p_{IDEAL}$.

(b) We now show the convergence claim, assuming Access Competence. By this assumption, there is an $\epsilon > 0$ such that $p_{s \to i} \geq 2^{-1/|S|} + \epsilon$ for all s and i. Consider a person i. The probability of having full source set S satisfies $Pr(\mathbf{S}_i = S) \geq \frac{1}{2} + \epsilon^{|S|}$, because

$$Pr(\mathbf{S}_i = S) = \prod_{s \in S} p_{s \to i} \geq \prod_{s \in S} (2^{-1/|S|} + \epsilon) = \left(2^{-1/|S|} + \epsilon\right)^{|S|} \geq (2^{-1/|S|})^{|S|} + \epsilon^{|S|} = \frac{1}{2} + \epsilon^{|S|}.$$

Since the full-access events '$\mathbf{S}_i = S$' $(i = 1, 2, ...)$ are mutually independent (by Independent Sources) and each of probability at least $\frac{1}{2} + \epsilon^{|S|}$, the probability that the proportion of members with full access exceeds $\frac{1}{2}$ (the event $\frac{\#\{i \in \{1,...,n\}: \mathbf{S}_i = S\}}{n} > \frac{1}{2}$) tends to one as $n \to \infty$, by the law of large numbers. In other words, the probability of a majority with full access (the event $\#\{i \in \{1, ..., n\} : \mathbf{S}_i = S\} > \frac{n}{2}$) tends to 1 as $n \to \infty$. Meanwhile, full access implies an ideal opinion (i.e., $\mathbf{S}_i = S$ implies $\mathbf{o}_i = \mathbf{o}_{IDEAL}$). So a majority with full access implies a majority with the ideal opinion (i.e., $\#\{i \in \{1, ..., n\} : \mathbf{S}_i = S\} > \frac{n}{2}$ implies $\mathbf{o}_{maj,n} = \mathbf{o}_{IDEAL}$). Hence, also the probability of an ideal majority opinion converges to one: $Pr(\mathbf{o}_{maj,n} = \mathbf{o}_{IDEAL}) \to 1$. This implies that $Pr(\mathbf{o}_{maj,n} = \mathbf{x}) \to Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$, i.e., that $p_{maj,n} \to p_{IDEAL}$. ■

Proof of the Post-Deliberation Jury Theorem. Assume a simple opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ and a share-absorb process, both for an infinite population $N = \{1, 2, ...\}$. The usual notation applies.

(a) The non-asymptotic claim holds by a version of the proof of part (a) of the Pre-Deliberation Jury Theorem. One should substitute $\mathbf{o}^+_{maj}$ for $\mathbf{o}_{maj}$, and apply the 'Claim' with $\mathbf{z} = (\mathbf{S}^+_i)$ rather than $\mathbf{z} = (\mathbf{S}_i)$, which is possible since also $(\mathbf{S}^+_i)$ is independent of $(\mathbf{x}, (\mathbf{e}_s))$.

(b) We now turn to the asymptotic claim. We shall face the difficulty of interpersonal correlations between post-deliberation source sets. The weak law of large numbers in Pivato's (2017) version for correlated variables will ultimately come to help, but first several claims must be established. We assume Acquisition Competence (needed only from Claim b5) and Non-Vanishing Participation (needed only from Claim b4).

Claim b1: For any source $s \in S$, group size $n \in \{1, 2, ...\}$, and group member $i \in \{1, ..., n\}$, the probability that another member shares s is

$$p_{s,i,n} = \overline{\prod_{j \in \{1,...,n\} \setminus \{i\}} \overline{p_{s \to j} p_{s,j \to}}}. \tag{12}$$

The probability is given by (12) because it equals the probability that it is *not* the case that each other member j does *not* share s, where j shares s with probability $p_{s \to j} p_{s,j \to}$, the product of the probabilities of accessing s and of sharing an accessed s. Q.e.d.

Claim b2: For any $s \in S$, $n \in \{1, 2, ...\}$, and $i \in \{1, ..., n\}$, the probability that some member other than i shares s and then i absorbs s, given that i has not accessed s initially, is $p_{s,i,n} p_{s,i \leftarrow}$.

The claim holds because the relevant probability is the product of the probability that someone else shares s, i.e., $p_{s,i,n}$ by Claim b1, and the probability that i absorbs a shared source s, i.e., $p_{s,i\leftarrow}$. Q.e.d.

Claim b3: For any $s \in S$, $n \in \{1, 2, ...\}$, and $i \in \{1, ..., n\}$, $Pr(s \in \mathbf{S}^+_{i,n}) = \overline{\overline{p_{s \to i}} \times \overline{p_{s,i,n} p_{s,i \leftarrow}}}$.

This holds because i does *not* hold s post-deliberation if and only if i does not initially access s (probability: $\overline{p_{s \to i}}$) and i does not absorb s (probability: $\overline{p_{s,i,n} p_{s,i\leftarrow}}$). Q.e.d.

Claim b4: For any $s \in S$ and $i \in \{1, 2, ...\}$, $P(s \in \mathbf{S}^+_{i,n}) \to \overline{\overline{p_{s \to i}} \times \overline{p_{s,i \leftarrow}}}$ as $n \to \infty$.

Fix s and i. By Claim b3, we just show $p_{s,i,n} \to 1$, i.e., $\prod_{j \in \{1,...,n\} \setminus \{i\}} \overline{p_{s \to j} p_{s,j \to}} \to 0$. By Non-Vanishing Participation, $p_{s \to j} p_{s,j \to} \not\to 0$ as $j \to \infty$, and hence $\overline{p_{s \to j} p_{s,j \to}} \not\to 1$ as $j \to \infty$. In consequence, $\prod_{j \in \{1,...,n\} \setminus \{i\}} \overline{p_{s \to j} p_{s,j \to}} \to 0$ as $n \to \infty$. Q.e.d.

Claim b5: For any $i \in \{1, 2, ...\}$, the full-access probability $Pr(\mathbf{S}^+_{i,n} = S)$ converges to a value of at least $\frac{1}{2} + \epsilon^{|S|}$ as $n \to \infty$, where $\epsilon > 0$ is the threshold in Acquisition Competence (which is independent of i).

Fix a person i. We have $Pr(\mathbf{S}^+_{i,n} = S) = \prod_{s \in S} Pr(s \in \mathbf{S}^+_{i,n})$, because the access events '$s \in \mathbf{S}^+_{i,n}$' are independent across sources s as a consequence of the fact that the pre-deliberation access events '$s \in \mathbf{S}_i$' are independent across sources (by Source Independence) and the share-absorb process operates independently across sources. So, by Claim b4, $Pr(\mathbf{S}^+_{i,n} = S) \to \prod_{s \in S} \overline{\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}}}$ as $n \to \infty$. Now choose $\epsilon > 0$ as in Acquisition Competence. Then, for all s, $\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}} \leq 1 - 2^{-1/|S|} - \epsilon$, i.e., $\overline{\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}}} \geq 2^{-1/|S|} + \epsilon$. So,

$$\prod_{s \in S} \overline{\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}}} \geq \left(2^{-1/|S|} + \epsilon\right)^{|S|} \geq (2^{-1/|S|})^{|S|} + \epsilon^{|S|} = \frac{1}{2} + \epsilon^{|S|}.$$

Hence $\lim_{n\to\infty} Pr(\mathbf{S}^+_{i,n} = S) \geq \frac{1}{2} + \epsilon^{|S|}$. Q.e.d.

Claim b6: For all $n \in \{1, 2, ...\}$ and distinct $i, j \in \{1, ..., n\}$, the covariance between i's and j's full access satisfies

$$Cov(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) \leq \prod_{s \in S} \prod_{k=i,j} \overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}} - \prod_{s \in S} \prod_{k=i,j} \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n} p_{s,k\leftarrow}}}$$

Fix $n \in \{1, 2, ...\}$ and distinct $i, j \in \{1, ..., n\}$. Then

$$\begin{aligned} Cov(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) &= Pr(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) - \prod_{k=i,j} Pr(\mathbf{S}^+_{k,n} = S) \\ &= \prod_{s \in S} Pr(s \in \mathbf{S}^+_{i,n}, s \in \mathbf{S}^+_{j,n}) - \prod_{s \in S} \prod_{k=i,j} Pr(s \in \mathbf{S}^+_{k,n}), \end{aligned}$$

where the second equality holds by independence across sources of the access events. Since $Pr(s \in \mathbf{S}^+_{k,n}) = \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n} p_{s,k\leftarrow}}}$ by Claim b3, it suffices to show that

$$Pr(s \in \mathbf{S}^+_{i,n}, s \in \mathbf{S}^+_{j,n}) \leq \prod_{k=i,j} \overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}} \text{ for all } s \in S.$$

This holds since, letting E be the event that s is shared by someone in $\{1, ..., n\}$,

$$\begin{aligned} Pr(s \in \mathbf{S}^+_{i,n}, s \in \mathbf{S}^+_{j,n}) &\leq Pr(s \in \mathbf{S}^+_{i,n}, s \in \mathbf{S}^+_{j,n} | E) \\ &= \prod_{k=i,j} Pr(s \in \mathbf{S}^+_{k,n} | E) = \prod_{k=i,j} \overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}}, \end{aligned}$$

where the first equality holds by independence between '$s \in \mathbf{S}^+_{i,n}$' and '$s \in \mathbf{S}^+_{i,n}$' given E, and the second equality holds because s is held ex-post if and only if it is *not* the case that s is *not* accessed ex-ante (probability: $\overline{p_{s\to k}}$) and *not* absorbed ex-post (probability: $\overline{p_{s,i\leftarrow}}$). Q.e.d.

Claim b7: : $\min_{s \in S, k \leq n} p_{s,k,n} \to 1$ as $n \to \infty$.

For all s and n, pick $i_{s,n} \in \{1, ..., n\}$ with $p_{s\to i_{s,n}} p_{s,i_{s,n}\to} = \max_{k \leq n} p_{s\to k} p_{s,k\to}$. By construction,

$$\min_{k \leq n} p_{s,k,n} = \overline{\prod_{j \in \{1,...,n\} \setminus \{i_{s,k}\}} \overline{p_{s\to j} p_{s,j\to}}}.$$

By Non-Vanishing Participation, $p_{s\to j}p_{s,j\to} \not\to 0$. So $\prod_{j\in\{1,\ldots,n\}\backslash\{i_{s,k}\}} \overline{p_{s\to j}p_{s,j\to}} \to 0$. Hence, $\min_{k\le n} p_{s,k,n} \to 1$. So, as $|S|$ is finite, $\min_{s\in S,k\le n} p_{s,k,n} \to 1$. Q.e.d.

Claim b8: $\delta_n \equiv \max_{s\in S,k\le n} \left(\overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}} - \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}}}\right) \to 0$ as $n \to \infty$.

For all $s \in S$, $n \in \{1, 2, ...\}$ and $k \le n$, we have

$$\begin{aligned}\overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}} - \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}}} &= \overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}} - \overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}} \\ &= \overline{p_{s\to k}} \left(\overline{p_{s,k,n}p_{s,k\leftarrow}} - \overline{p_{s,k\leftarrow}}\right) \\ &= \overline{p_{s\to k}} \left(p_{s,k\leftarrow} - p_{s,k,n}p_{s,k\leftarrow}\right) \\ &= \overline{p_{s\to k}}p_{s,k\leftarrow} \left(1 - p_{s,k,n}\right) \\ &\ge 1 - p_{s,k,n}.\end{aligned}$$

This lower bound implies the desired convergence via Claim b7. Q.e.d.

Claim b9: The average covariance of full access between group members converges to zero, i.e.,

$$\frac{1}{n^2}\sum_{i,j=1}^{n} Cov(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) \to 0 \text{ as } n \to \infty.$$

It suffices to show that $Cov(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) \le 1$ whenever $i = j$ $(\le n)$ and that $\max_{i,j\le n,i\ne i} Cov(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) \to 0$, by a simple argument (which uses that each $Cov(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S)$ is positive). The former is obvious. We now show the latter. By Claim b6 and the positivity of all the covariances, it is enough to prove that, for any distinct i, j,

$$\prod_{s\in S}\prod_{k=i,j} a_{s,k} - \prod_{s\in S}\prod_{k=i,j} a_{s,k,n} \to 0 \text{ as } n \to \infty,$$

where $a_{s,k} = \overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}}$ and $a_{s,k,n} = \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}}}$. Fix distinct i, j. Note that $a_{k,s} = (a_{s,k} - a_{s,k,n}) + a_{s,k,n} \le \delta_n + a_{s,k,n}$, by Claim b8. So it suffices to show that

$$\prod_{s\in S}\prod_{k=i,j} (\delta_n + a_{s,k,n}) - \prod_{s\in S}\prod_{k=i,j} a_{s,k,n} \to 0 \text{ as } n \to \infty. \tag{13}$$

By developing the product $\prod_{s\in S}\prod_{k=i,j}(\delta_n + a_{s,k,n})$, check that it equals a polynomial in δ_n (of order $2|S|$) whose constant term is '$+\prod_{s\in S}\prod_{k=i,j} a_{s,k,n}$'. This constant term cancels out against '$-\prod_{s\in S}\prod_{k=i,j} a_{s,k,n}$', so that the expression in (13) is a polynomial in δ_n with *zero* constant term. As $n \to \infty$, δ_n converges to 0 (by Claim b8), and so any polynomial in δ_n with zero constant term also converges to 0. This proves (13). Q.e.d.

Claim b10: $p^+_{maj,n} \to p_{IDEAL}$. (This completes the proof.)

Since every person i's full-access event $\mathbf{S}^+_{i,n} = S$ has probability converging to $\frac{1}{2} + \epsilon^{|S|}$ as $n \to \infty$ by Claim b5, while the average covariance of these events tends to zero by Claim b9, the probability that the proportion of members with full access exceeds $\frac{1}{2}$ (the event $\frac{\#\{i\in\{1,\ldots,n\}:\mathbf{S}^+_{i,n}=S\}}{n} > \frac{1}{2}$) tends to one, by the weak law of large numbers in Pivato's (2017) version for correlated variables.[21] Equivalently, the probability of a majority with full access (the event $\#\{i \in \{1, ..., n\} : \mathbf{S}^+_{i,n} = S\} > \frac{n}{2}$) tends to 1. Since (a majority with) full access implies (a majority with) an ideal opinion, also the probability of an ideal majority opinion converges to 1: $Pr(\mathbf{o}^+_{maj,n} = \mathbf{o}_{IDEAL}) \to 1$. So, $Pr(\mathbf{o}^+_{maj,n} = \mathbf{x}) \to Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$, i.e., $p^+_{maj,n} \to p_{IDEAL}$. ■

[21] This version of the law follows from the proof of Pivato's Theorem 5.2 (i.e., from Claim 2 in that proof, combined with Chebyshev's Inequality). A closely related result is Proposition A2 in Pivato (2016).