Judgment aggregation under constraints

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In solving judgment aggregation problems, groups often face constraints. Many decision problems can be modelled in terms the acceptance or rejection of certain propositions in a language, and constraints as propositions that the decisions should be consistent with. For example, court judgments in breach-of-contract cases should be consistent with the constraint that action and obligation are necessary and sufficient for liability; judgments on how to rank several options in an order of preference with the constraint of transitivity; and judgments on budget items with budgetary constraints. Often more or less demanding constraints on decisions are imaginable. For instance, in preference ranking problems, the transitivity constraint is often constrasted with the weaker acyclicity constraint. In this paper, we make constraints explicit in judgment aggregation by relativizing the rationality conditions of consistency and deductive closure to a constraint set, whose variation yields more or less strong notions of rationality. We review several general results on judgment aggregation in light of such constraints.

1 Introduction

There has been much recent work on the problem of 'judgment aggregation': How can the judgments of several individuals on logically connected propositions be aggregated into corresponding collective judgments (e.g., List and Pettit 2002, Pauly and van Hees 2006, Dietrich 2006, Nehring and Puppe forthcoming)? To illustrate, consider the much-cited example of the 'doctrinal paradox' (Kornhauser and Sager 1986). Suppose a three-member court has to make collective judgments (acceptance/rejection) on three connected propositions:

a: The defendant did action X.

b: The defendant had a contractual obligation not to do action X.

c: The defendant is liable for breach of contract.

Suppose further that legal doctrine imposes the constraint that action and obligation (the two *premises*) are necessary and sufficient for liability (the *conclusion*), in short $c \leftrightarrow (a \wedge b)$. It can then happen that the majority judgments on the two premises (a and b) conflict with the majority judgment on the conclusion (c), relative to that constraint. Suppose, for example, the first judge holds both a and b to be true; the second holds a but not b to be true; and the third holds b but not a to be true. If each judge individually respects the constraint that $c \leftrightarrow (a \wedge b)$, then the majority judgments – in support of a and b and against c – violate the given constraint, as shown in Table 1.

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	a	b	c
Individual 1	True	True	True
Individual 2	True	False	False
Individual 3	False	True	False
Majority	True	True	False

Table 1: The doctrinal paradox

The conflict may disappear if we modify the constraint. For example, the majority judgments $\{a, b, \neg c\}$ pose no problem if a and b are considered necessary but not sufficient for liability (so that the constraint is $c \rightarrow (a \wedge b)$ instead of $c \leftrightarrow (a \wedge b)$), or if we introduce a third premise d (so that the constraint is $c \leftrightarrow (a \wedge b \wedge d)$), or if we drop the constraint altogether.

Our aim in this paper is to investigate judgment aggregation on general agendas of propositions with general sets of constraints. This framework is suitable for modelling not only the court example but also many other judgment aggregation problems. Judgments on budget items, for example, are required to respect budgetary constraints. If propositions a, b and c state, respectively, that spending on education, healthcare and defense should be increased, then a budgetary constraint could stipulate that not all three can be accepted together, formally $\neg(a \wedge b \wedge c)$. Judgments on binary ranking propositions such as 'x is preferable to y', 'y is preferable to z' and 'x is preferable to z' are connected by constraints such as transitivity or acyclicity. Judgments of biologists on whether two organisms fall into the same species are constrained by the assumption that belonging to the same species is an equivalence relation.

We explain how constraints between propositions can be naturally incorporated into the judgment aggregation model. Constraints have of course played a role in earlier work, particularly in the computer science literature under the label 'integrity constraints' (e.g., Konieczny and Pino-Perez 2002). See also the notion of 'context' in Nehring and Puppe (forthcoming) and that of the 'axioms' in Dietrich (2007).

We present two general impossibility theorems that depend on the nature of those constraints. The results are corollaries of results in Dietrich and List (2007a), but have a somewhat different interpretational flavour. They are also closely related to results by Dokow and Holzman (2005) and prior results by Nehring and Puppe (2002).

To illustrate our approach, we apply our two theorems to the aggregation of judgments on binary relations (which can represent various forms of comparisons), distinguishing between different constraint sets on such binary relations. In particular, we consider strict orderings, acyclic binary relations and equivalence relations. This application generalizes earlier results by List and Pettit (2001/2004), Dietrich (2007), Dietrich and List (2007a) and Nehring and Puppe (forthcoming) on the representation of preference aggregation in the judgment aggregation model (a related result drawing on the 'property space' framework is Nehring 2003). A comprehensive bibliography on judgment aggregation can be found online (List 2004-7).

2 The model

We consider a group of individuals $N = \{1, 2, \ldots, n\}$ ($n \geq 2$). The propositions on which judgments are made are represented in logic (following List and Pettit 2002, 2004; we use Dietrich's 2007 generalized model).

Logic. A *logic* is an ordered pair $(\mathbf{L}, \vdash)$, where (i) $\mathbf{L}$ is a non-empty set of sentences, called *propositions*, closed under negation (i.e., if $p \in \mathbf{L}$ then $\neg p \in \mathbf{L}$, where $\neg$ denotes 'not'), and (ii) $\vdash$ is an *entailment relation*, where, for each set $S \subseteq \mathbf{L}$ and each proposition $p \in \mathbf{L}$, $S \vdash p$ is read as 'S entails p' (we write $p \vdash q$ to abbreviate $\{p\} \vdash q$).[2] A set $S \subseteq \mathbf{L}$ is *inconsistent* if $S \vdash p$ and $S \vdash \neg p$ for some $p \in \mathbf{L}$, and *consistent* otherwise. We require the logic to satisfy the following minimal conditions:[3]

(L1) For all $p \in \mathbf{L}$, $p \vdash p$ (self-entailment).

(L2) For all $p \in \mathbf{L}$ and $S \subseteq T \subseteq \mathbf{L}$, if $S \vdash p$ then $T \vdash p$ (monotonicity).

(L3) $\emptyset$ is consistent, and each consistent set $S \subseteq \mathbf{L}$ has a consistent superset $T \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$ (completability).

In standard propositional logic, $\mathbf{L}$ contains propositions such as a, b, $a \wedge b$, $a \vee b$, $\neg(a \rightarrow b)$ (where $\wedge$, $\vee$, $\rightarrow$ denote 'and', 'or', 'if-then', respectively). The set $\{a, a \rightarrow b\}$ entails proposition b, for example, whereas the set $\{a \vee b\}$ does not entail a. Examples of consistent sets are $\{a, a \rightarrow b, b\}$ and $\{a \wedge b\}$, examples of inconsistent ones $\{a, \neg a\}$ and $\{a, a \rightarrow b, \neg b\}$.

Agenda. The *agenda* is the set of propositions on which judgments are made, defined as a non-empty subset $X \subseteq \mathbf{L}$ expressible as $X = \{p, \neg p : p \in X_+\}$ for a set $X_+ \subseteq \mathbf{L}$ of unnegated propositions. Notationally, we assume that double negations cancel each other out (i.e., $\neg\neg p$ stands for p).[4] In the three-member court example, $X = \{a, \neg a, b, \neg b, c, \neg c\}$.

[2] Formally, $\vdash \subseteq \mathcal{P}(\mathbf{L}) \times \mathbf{L}$, where $\mathcal{P}(\mathbf{L})$ is the power set of $\mathbf{L}$.

[3] Alternatively we may assume three conditions on the consistency notion (jointly equivalent to L1-L3): (C1) All sets $\{p, \neg p\} \subseteq \mathbf{L}$ are inconsistent; (C2) subsets of consistent sets $S \subseteq \mathbf{L}$ are consistent; (C3) L3 holds. In many (non-paraconsistent) logics, the notion of entailment is uniquely determined by that of consistency (via $A \vdash p \Leftrightarrow [A \cup \{\neg p\}$ is inconsistent$]$), so that the two notions are interdefinable. If we restrict attention to logics with interdefinability, or if we are ultimately interested only in whether judgments are consistent (not in whether they are deductively closed), we can use the system of consistent sets rather than the relation $\vdash$ as the primitive logical notion (and assume C1-C3). For details see Dietrich (2007).

[4] Strictly speaking, when we use the symbol $\neg$ hereafter, we mean a modified negation symbol $\sim$, where $\sim p := \neg p$ if p is unnegated and $\sim p := q$ if $p = \neg q$ for some q. This convention is to ensure that $p \in X$ implies $\neg p \in X$.

Constraints. A *constraint set* is a consistent subset $C \subseteq \mathbf{L}$. It is meant to represent logical interconnections that are stipulated to hold between propositions. In the three-member court example, $C = \{c \leftrightarrow (a \wedge b)\}$. We say that a set $S \subseteq \mathbf{L}$ *entails* a proposition $p \in \mathbf{L}$ *relative to* C, formally $S \vdash_C p$, if $S \cup C \vdash p$. We say that a set $S \subseteq \mathbf{L}$ is *consistent relative to* C if $S \cup C$ is consistent, and *inconsistent relative to* C otherwise. Hereafter we refer to *C-entailment* and *C-(in)consistency*. The relationship between C-(in)consistency and C-entailment is analogous to that between (in)consistency and entailment *simpliciter*, which can be seen as the special cases of C-(in)consistency and C-entailment for $C = \emptyset$. A set $S \subseteq \mathbf{L}$ is *minimally C-inconsistent* if S is C-inconsistent but every proper subset of S is C-consistent. A proposition $p \in \mathbf{L}$ is *C-contingent* if $\{p\}$ and $\{\neg p\}$ are C-consistent. Informally, a C-contingent proposition is one whose truth or falsity is not settled by the constraints in C alone.

Individual judgment sets. Each individual i's judgment set is the set $A_i \subseteq X$ of propositions that he or she accepts. On a belief interpretation, A_i is the set of propositions believed by individual i to be true; on a desire interpretation, the set of propositions desired by individual i to be true. A judgment set A_i is

- *C-consistent* if, as just defined, $A_i \cup C$ is consistent;
- *C-deductively closed* if it contains all propositions $p \in X$ such that $A_i \cup C \vdash p$ (i.e., $A_i \vdash_C p$);
- *complete* if it contains a member of each proposition-negation pair $p, \neg p \in X$.

A *profile* is an n-tuple $(A_1, \ldots, A_n)$ of individual judgment sets.

Aggregation functions. An *aggregation function* is a function F that maps each profile $(A_1, \ldots, A_n)$ from some domain of admissible ones to a collective judgment set $F(A_1, \ldots, A_n) = A \subseteq X$, the set of propositions that the group as a whole accepts. The judgment set A can be interpreted as the set of propositions collectively believed to be true or as the set collectively desired to be true. Below we impose minimal conditions on aggregation functions (including on the domain of admissible profiles and the co-domain of admissible collective judgment sets). Standard examples of aggregation functions are

- *majority voting*, where $F(A_1, ..., A_n)$ is the set of propositions $p \in X$ for which the number of individuals with $p \in A_i$ exceeds that with $p \notin A_i$;
- *dictatorships*, where $F(A_1, ..., A_n) = A_i$ for some antecedently fixed individual $i \in N$; and
- *inverse dictatorships*, where $F(A_1, ..., A_n) = \{\neg p : p \in A_i\}$ for some antecedently fixed individual $i \in N$.

3 Why explicit constraints?

We could avoid explicit reference to constraints by building them into the logic. Indeed, whenever the logic $(\mathbf{L}, \vdash)$ satisfies L1, L2 and L3, then so does the logic $(\mathbf{L}, \vdash_C)$ induced by the constraint set C. C-consistency in $(\mathbf{L}, \vdash)$ translates into standard consistency in $(\mathbf{L}, \vdash_C)$, and C-deductive closure in $(\mathbf{L}, \vdash)$ translates into standard deductive closure in $(\mathbf{L}, \vdash_C)$. This is in fact the only insight needed to translate existing theorems into theorems with explicit constraints.

Why, then, should we use explicit constraints at all? First of all, constraints introduce a different perspective on the notion of consistency. For a judgment set to be logically inconsistent is somewhat different and perhaps more dramatically 'irrational' than to be merely C-inconsistent, i.e., incompatible with the given constraints. If constraints are built into the logic, the distinction between these two kinds of inconsistency disappears: all inconsistencies are by definition logical ones.

Second, the nature of the appropriate set of constraints is often unclear or controversial. For example, what are the correct budgetary constraints or legal constraints when a government cabinet makes decisions? It may thus be interesting to vary the constraint set C, so that we can express the fact that a judgment set is C-consistent yet C'-inconsistent (for distinct $C, C' \subseteq \mathbf{L}$). If reaching C-consistent collective judgments turns out to be unrealistic, the group might look for C'-consistent collective judgments for a 'less ambitious' constraint set C', say a proper subset $C' \subsetneq C$. There is a long tradition in social choice theory of considering differently strong rationality constraints on preferences: one may or may not require completeness, one may or may not require full transitivity etc. As discussed later, each set of rationality conditions on preferences corresponds to a particular constraint set.

Third, if it is unclear for some proposition $p \in \mathbf{L}$ whether or not it should constrain the group decision, a natural move is to put it into the agenda X (rather than into the constraint set C): i.e., to let the group decide whether or not p should constrain the judgments on the (other) propositions in the agenda. For instance, the 'legal doctrine' in the introductory court example or the condition of a balanced budget might be made part of the agenda X rather than of the constraint set C.

When a constraint becomes a proposition under decision, its correct logical representation becomes crucial. Let us illustrate this point using the two examples just mentioned. First, consider the court example, and suppose the 'legal doctrine' (that action and obligation are necessary and sufficient for liability) is not imposed on the judges but put up for decision. One might be tempted to represent the legal doctrine as a material biconditional $c \leftrightarrow (a \wedge b)$. This, however, is a problematic representation. Consider the resulting agenda $X = \{a, \neg a, b, \neg b, c, \neg c, c \leftrightarrow (a \wedge b), \neg(c \leftrightarrow (a \wedge b))\}$. When a judge rejects the legal doctrine, what he or she rejects is in fact not the material biimplication $c \leftrightarrow (a \wedge b)$; indeed, he or she may well believe that a, b and c are all true or all false (so that $c \leftrightarrow (a \wedge b)$ holds). Rather the judge rejects the binding nature of a and b for c. One might say, the judge rejects a *subjunctive* biconditional between c and $a \wedge b$, or perhaps that he or she rejects the proposition $\blacksquare(c \leftrightarrow (a \wedge b))$, where $\blacksquare$ is a modal necessity operator ('necessarily, i.e., in all possible worlds, it is the case that...'). If the legal doctrine is represented using a subjunctive biconditional or modal necessity operator, negating the resulting proposition becomes logically consistent with assigning arbitrary truth values to a, b and c, so that the previous problem is avoided.

Similarly, suppose a government faces a decision problem, and suppose a balanced budget is not imposed as a constraint but represented by a proposition p in the government's agenda X. One might be tempted to specify p as the disjunction $\vee_{q \in S} q$, where each proposition $q \in S$ describes a way in which the budget can be balanced (such as 'low spending on education and average spending on social security and...'). The problem here is similar to that just identified in the court example. An individual who rejects the requirement that the budget *must* be balanced may still hold other beliefs that entail a balanced budget (i.e., that entail $\vee_{q \in S} q$): he or she may see no *necessity* of a balanced budget yet favour low total spending for other reasons. A more appropriate representation of the balanced budget requirement might be to let p be the proposition $O(\vee_{q \in S} q)$, where O is a deontic 'ought' operator ('it is required that...'). Since $O(\vee_{q \in S} q)$ states that the budget *ought* to be balanced, it becomes consistent (in standard deontic logic) to negate $O(\vee_{q \in S} q)$ while asserting $\vee_{q \in S} q$, which removes the problem that arises when p is defined as $\vee_{q \in S} q$.

However, if a constraint is not part of the agenda but part of the constraint set C, its misrepresentation is less problematic. The reason is that propositions in C cannot be negated, and often the logical interconnections induced by the (non-negated) constraints in the form of C-consistency and C-deductive closure do not change if these constraints are misspecified in the sense just illustrated. In the court example, for instance, the material biimplication $c \leftrightarrow (a \wedge b)$ imposes exactly the same constraints on a, b and c as a subjunctive one, and also as the proposition $\blacksquare(c \leftrightarrow (a \wedge b))$, namely that a, b, c can only have truth values (T, T, T) or (F, F, F) or (T, F, F) or (F, T, F). For this reason, when giving concrete examples of constraint sets in this paper we usually omit modal or deontic necessity operators and do not address the nature of (bi)conditionals. For instance, when we later consider the transitivity constraint on preferences, we model it as the statement that 'preferences *are* transitive', not the statement that 'preferences *are necessarily* transitive' (and similarly for other constraints on preferences).[5]

[5] On subjunctive implications in judgment aggregation, see Dietrich (forthcoming); on modal operators for representing legal prescriptions, see List (2006) and Dietrich (2007).

4 Impossibility results

Can we find attractive aggregation functions? The answer to this question depends on two things. First, it depends on what conditions we impose on the aggregation function. If, for example, we do not seek to achieve C-consistency at the collective level (for an appropriate C), majority voting may be a perfectly fine solution. Likewise, in the absence of any democratic requirements, a dictatorship of one individual arises as a possibility, which generates C-consistent and complete judgment sets. If the only democratic requirement is non-dictatorship and we allow collective judgments to be incomplete but retain their C-consistency and C-deductive closure, then oligarchies arise as a solution; here, any proposition is accepted if and only if all members of a fixed set $M \subseteq N$ of 'oligarchs' accept it.[6]

Second, the question of whether we can find attractive aggregation functions depends on how the propositions in the agenda are logically connected, which in turn depends on the constraint set C. More constraints can often make aggregation problems harder to solve. If the court in the original example did not have to respect the constraint that action and obligation are necessary and sufficient for liability ($c \leftrightarrow (a \wedge b)$), then the majority judgments resulting from the individual judgments in Table 1 would not be considered inconsistent.

Let us address these questions in general terms. Consider some given agenda X and constraint set C. The theorems to be presented here are C-relativized versions of existing theorems from Dietrich and List (2007a). We choose to focus on theorems that require complete and C-consistent (hence also C-deductively closed) collective judgment sets. But one could equally obtain theorems that require merely C-consistent and C-deductively closed (possibly incomplete) collective judgment sets (by adapting results by Dietrich and List forthcoming and Dokow and Holzman 2006), or theorems that require just C-consistent collective judgment sets (by adapting recent results by Dietrich and List 2007b).

More precisely, we here require the aggregation function to satisfy the following conditions:

Universal C-domain. The domain of F is the set of all possible profiles of C-consistent and complete individual judgment sets on the agenda X.

Collective C-rationality. F generates C-consistent and complete collective judgment sets on the agenda X.

[6] More precisely, an oligarchy F is defined by $F(A_1, .., A_n) = \cap_{i \in M} A_i$ for all profiles $(A_1, ..., A_n)$ in the universal C-domain (as defined below), where $M \subseteq N$ is a fixed non-empty set of 'oligarchs'. Oligarchies generate C-consistent and C-deductively closed (but usually incomplete) collective judgment sets, as the intersection of C-consistent and C-deductively closed sets is C-consistent and C-deductively closed. To avoid dictatorship, there must be at least two oligarchs; if all individuals are oligarchs, F is unanimity rule, an anonymous rule with considerable collective incompleteness.

Systematicity. For any propositions $p, q \in X$ and profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all individuals i, $p \in A_i$ if and only if $q \in A_i^*$] then [$p \in F(A_1, \ldots, A_n)$ if and only if $q \in F(A_1^*, \ldots, A_n^*)$].

Universal C-domain requires that the aggregation function accept as admissible any possible profile of fully rational individual judgment sets respecting the constraints in the set C. Collective C-rationality requires that the aggregation function produce as output a fully rational collective judgment set respecting the same constraints. Systematicity requires, first, that the collective judgment on each proposition depend only on individual judgments on that proposition and, second, that the pattern of dependence be the same for all propositions. The first part of the condition is the *independence* part, the second the *neutrality* part.

Call agenda X *minimally C-connected* if it satisfies the following conditions:

(i) X has a minimal C-inconsistent subset Y with $|Y| \geq 3$, and

(ii) X has a minimal C-inconsistent subset Y such that $(Y \backslash Z) \cup \{\neg z : z \in Z\}$ is C-consistent for some subset $Z \subseteq Y$ of even size.[7]

It is easy to see that the agenda $X = \{a, \neg a, b, \neg b, c, \neg c\}$ in the three-member court example with constraint set $C = \{c \leftrightarrow (a \wedge b)\}$ satisfies minimal C-connectedness. On the other hand, if C were the empty set, the agenda $X = \{a, \neg a, b, \neg b, c, \neg c\}$ would not be minimally C-connected: it would violate both (i) and (ii). Thus the question of whether or not an agenda is minimally C-connected depends crucially on the strength of the constraint set C.

The following is a corollary of Dietrich and List's (2007) Theorem 1 (which in turn generalizes earlier results on systematicity by List and Pettit 2002 and Pauly and van Hees 2006):

Theorem 1. For a minimally C-connected agenda X, every aggregation function F satisfying universal C-domain, collective C-rationality and systematicity is a (possibly inverse) dictatorship.

The agenda condition of Theorem 1 (minimal C-connectedness) is tight if the agenda is finite or the logic is compact (and $n \geq 3$ and X contains at least one C-contingent proposition), i.e., minimal C-connectedness is also necessary, and not merely sufficient, for characterizing (possibly inverse) dictatorships by the conditions of Theorem 1.[8] The same holds for the agenda conditions of the other theorems stated below.

[7]This clause is for finite X equivalent to a C-relativized version of Dokow and Holzman's (2005) *non-affineness* condition: the set of admissible yes/no views on the propositions in X (corresponding to C-consistent and complete judgment sets on X) is a non-affine subset of $\{0,1\}^X$.

[8]If X is not minimally C-connected, there exists an aggregation function that satisfies universal C-domain, collective C-rationality and systematicity and is not a (possibly inverse)

There are two ways in which Theorem 1 can be turned into a characterization of dictatorships as opposed to possibly inverse ones. One way is to impose an additional unanimity condition on the aggregation function:

Unanimity. For any unanimous profile $(A, \ldots, A) \in Domain(F)$, $F(A, \ldots, A) = A$.

Theorem 1a. For a minimally C-connected agenda X, every aggregation function F satisfying universal C-domain, collective C-rationality, systematicity and unanimity is a dictatorship.

The other way to obtain a characterization of dictatorships from Theorem 1 is to impose an additional asymmetry condition on the agenda. Call agenda X *C-asymmetric* if there exists a C-inconsistent subset $Y \subseteq X$ such that $\{\neg y : y \in Y\}$ is C-consistent.

Theorem 1b. For a minimally C-connected and C-asymmetric agenda X, every aggregation function F satisfying universal C-domain, collective C-rationality and systematicity is a dictatorship.

Systematicity, however, is a strong condition on an aggregation function, and it is interesting to ask whether we can obtain a characterization of dictatorships using the weaker condition of independence, which retains the independence part of systematicity but drops the neutrality part.

Independence. For any proposition $p \in X$ and profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all individuals i, $p \in A_i$ if and only if $p \in A_i^*$] then [$p \in F(A_1, \ldots, A_n)$ if and only if $p \in F(A_1^*, \ldots, A_n^*)$].

Let us define the agenda condition of *C-path-connectedness*, building upon Nehring and Puppe's (2002) condition of *total blockedness*.[9] For any $p, q \in X$, we write $p \vdash_C^* q$ if $\{p, \neg q\} \cup Y$ is C-inconsistent for some $Y \subseteq X$ that is C-consistent with p and with $\neg q$.[10] Now an agenda X is *C-path-connected* if

(iii) for every C-contingent $p, q \in X$, there exist $p_1, p_2, ..., p_k \in X$ (with $p = p_1$ and $q = p_k$) such that $p_1 \vdash_C^* p_2$, $p_2 \vdash_C^* p_3$, ..., $p_{k-1} \vdash_C^* p_k$.

dictatorship. Let M be a subset of $\{1, ..., n\}$ of odd size at least 3. If part (i) of minimal C-connectedness is violated, then majority voting among the individuals in M satisfies all requirements. If part (ii) is violated, the aggregation rule F with universal C-domain defined by $F(A_1, ..., A_n) := \{p \in X :$ the number of individuals $i \in M$ with $p \in A_i$ is odd$\}$ satisfies all requirements. The second example is based on Dokow and Holzman (2005).

[9] The relationship between C-path-connectedness and total blockedness arises when $C = \varnothing$. For a compact logic, $\varnothing$-path-connectedness is equivalent to total blockedness; generally, $\varnothing$-path-connectedness is weaker than total blockedness.

[10] For non-paraconsistent logics (in the sense of L4 in Dietrich 2007), $\{p, \neg q\} \cup Y$ is C-inconsistent if and only if $\{p\} \cup Y \vdash_C q$.

The agenda in the three-member court example above is minimally C-connected but not C-path-connected, but as shown below, preference aggregation problems can be represented by agendas that are both minimally C-connected and C-path-connected. Call an agenda *strongly C-connected* if it is C-path-connected and satisfies (ii). It then follows (for finite X or a compact logic) that X also satisfies (i) and hence that it is minimally C-connected as well.

Theorem 2. For a strongly C-connected agenda X, every aggregation function F satisfying universal C-domain, collective C-rationality, independence and unanimity is a dictatorship.

This result is the C-relativized version of a result proved independently by Dietrich and List (2007) and Dokow and Holzman (2005).[11] Both of these results extend a prior result by Nehring and Puppe (2002) with an additional monotonicity condition on F.

Finally, all results in this section continue to hold under generalized definitions of minimal and strong C-connectedness.[12]

5 An application: binary relations

To illustrate the results above, we apply them to the aggregation of binary comparisons, such as betterness judgments or judgments of (a given type of) equivalence. Such judgments are given by a binary relation over a set of objects to be compared, e.g., policy alternatives, job candidates or organisms to be classified into species. How can binary relations be represented in the judgment aggregation model? We use the following construction, drawing on List and Pettit (2001/2004), Dietrich (2007) and Dietrich and List (2007).

A simple predicate logic. We consider a predicate logic with constants $x, y, z, ... \in K$ (representing objects), variables v, w, v_1, v_2 , ... (ranging over objects), identity symbol $=$, a binary relation symbol P (representing the comparative relation in question), logical connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (if-then), and universal quantifier $\forall$. Formally, $\mathbf{L}$ is the smallest set such that

[11] Dokow and Holzman restrict the agenda to be finite (with only contingent propositions) and for this case show the tightness of the agenda assumptions (if $n \geq 3$).

[12] In the definitions of minimal and strong C-connectedness, (i) and (ii) can be weakened, namely to the C-relativised versions of the conditions (i*) and (ii*) given in Dietrich (2007). All theorems presented survive the weakening, and the agenda assumptions of Theorems 1, 1a and 1b become tight even for infinite X in a non-compact logic (again provided that X contains a contingent proposition and $n \geq 3$). The weakened conditions become equivalent to the original ones for finite X or a compact logic.

- **L** contains all propositions of the forms $\alpha P \beta$ and $\alpha = \beta$, where α and β are constants or variables, and
- whenever **L** contains two propositions p and q, then **L** also contains $\neg p$, $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$ and $(\forall v)p$, where v is any variable.

We drop brackets when there is no ambiguity.

Constraint sets. We consider some alternative constraint sets. We begin with the constraint set on fully rational strict preferences, the paradigmatic binary relation in social choice theory:

$$C_{\text{fully rational}} = \left\{ \begin{array}{l} (\forall v_1)(\forall v_2)(v_1 P v_2 \rightarrow \neg v_2 P v_1) \\ (\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \rightarrow v_1 P v_3) \\ (\forall v_1)(\forall v_2)(\neg\, v_1{=}v_2 \rightarrow (v_1 P v_2 \vee v_2 P v_1)) \end{array} \right\}^{13}.$$

The three displayed propositions in $C_{\text{fully rational}}$ are the constraints of asymmetry, transitivity and connectedness. To represent weak preferences rather than strict ones, $C_{\text{fully rational}}$ needs to be redefined as the set of rationality conditions on weak preferences (i.e., reflexivity, transitivity and connectedness); see also Dietrich (2007).[14]

Contrast this with the constraint set on merely acyclic (but not necessarily fully rational) strict preferences, representing a weaker notion of rationality:

$$C_{\text{acyclic}} = \left\{ \begin{array}{r} \neg(\alpha_1 P a_2 \wedge ... \wedge \alpha_{m-1} P a_m \wedge a_m P a_1) \\ : a_1, ...a_m \in K \text{ pairwise distinct, } m \geq 1 \end{array} \right\}^{15}.$$

The propositions in C_{acyclic} rule out any cycle of any length $m \geq 1$. In particular, irreflexivity is enforced (take $m = 1$). Transitivity, however, is not required. Thus the set $\{xPy, yPz, \neg xPz\}$, while inconsistent relative to $C_{\text{fully rational}}$, is consistent relative to C_{acyclic}.

Next we consider the constraint set on equivalence relations, suitable for classifying objects:

$$C_{\text{equivalence}} = \left\{ \begin{array}{l} (\forall v)(vPv) \\ (\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \rightarrow v_1 P v_3) \\ (\forall v_1)(\forall v_2)(v_1 P v_2 \rightarrow v_2 P v_1)) \end{array} \right\}^{16}.$$

The three displayed propositions in $C_{\text{equivalence}}$ are the constraints of reflexivity, transitivity and symmetry. While this constraint set would obviously not be

[13] For technical reasons, the constraint set also contains, for each pair of distinct constants x, y, the condition $\neg\, x{=}y$.

[14] Transitivity and connectedness are as defined above. Reflexivity can be stated by the proposition $(\forall v)(vPv)$. For aesthetic reasons, one might also replace the predicate symbol P by R in the logic.

[15] Again, the constraint set also contains, for each pair of distinct constants x, y, the condition $\neg\, x{=}y$.

[16] Again, the constraint set also contains, for each pair of distinct constants x, y, the condition $\neg\, x{=}y$.

imposed when P represents a *preference* relation (since 'better than' is neither reflexive nor symmetric), it may be imposed on a relation of *equal suitability* between job candidates (since 'is as suitable as' is plausibly an equivalence relation) or on the relation of *belonging to the same species* among organisms.

Each of these constraint sets C induces its own notions of C-consistency and C-deductive closure.

The agenda. The *binary-relation agenda* is the set X of all propositions of the form $xPy, \neg xPy \in \mathbf{L}$, where x and y are constants. The question of which agenda condition is met by the binary-relation agenda depends crucially on the given constraint set. The following lemma holds:

Lemma 1. The binary-relation agenda X (with $|K| \geq 3$) is
(a) strongly C-connected when $C = C_{\text{fully rational}}$;
(b) minimally, but not strongly, C-connected when $C = C_{\text{acyclic}}$;
(c) minimally, but not strongly, C-connected when $C = C_{\text{equivalence}}$.

In part (a), the C-path-connectedness part is a variant of a lemma by Nehring (2003); for instance, $xPy \vdash_C^* xPz$ because $\{xPy, yPz\} \vdash_C xPz$ (where $x, y, z \in K$ are pairwise distinct). In parts (a) and (b), minimal C-connectedness holds since any cycle $Y = \{xPy, yPz, zPx\} \subseteq X$ defines a minimal C-inconsistent set, which becomes C-consistent by negating two elements. In part (c), minimal C-connectedness holds because any set of type $Y = \{xPy, yPz, \neg xPz\} \subseteq X$ (with x, y, z pairwise distinct) is minimally C-inconsistent and becomes C-consistent by negating any two members.

By this lemma, Theorems 1, 1a and 1b apply to the binary relation agenda for any of the three constraint sets C. This allows the conclusion that it is impossible to aggregate preference relations – whether fully rational or just acyclic – or equivalence relations in a systematic and non-degenerate way, unless we restrict the domain of individual inputs or allow some kind of collective irrationality (such as incomplete collective judgment sets).

By part (a), the stronger impossibility of Theorem 2 applies when the constraint set is $C_{\text{fully rational}}$. It is impossible to aggregate fully rational preference relations in an independent, unanimity preserving and non-dictatorial manner, again unless we restrict the domain of individual inputs or allow collective irrationality. The latter is precisely Arrow's famous theorem on the aggregation of preferences (in the case where indifference between distinct options is excluded).

In conclusion, the present approach allows us to derive a large number of general results on aggregation problems with various constraints in a simple unified framework. An interesting question for future research is how the results are affected when different constraints are imposed at individual and collective levels, for example, when the constraints on collective judgments are weaker than those on individual ones or vice-versa.

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