Bargaining with endogenous disagreement: the extended Kalai-Smorodinsky solution*

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Abstract

Following Vartiainen (2007) we consider bargaining problems in which no exogenous disagreement outcome is given. A bargaining solution assigns a pair of outcomes to such a problem, namely a compromise outcome and a disagreement outcome: the disagreement outcome may serve as a reference point for the compromise outcome, but other interpretations are given as well. For this framework we propose and study an extension of the classical Kalai-Smorodinsky bargaining solution. We identify the (large) domain on which this solution is single-valued, and present two axiomatic characterizations on subsets of this domain.

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1 Introduction

In the bargaining problem of Nash (1950) each player can unilaterally enforce the disagreement outcome if negotiations fail. In some cases, however, it may not be clear what the disagreement outcome is or whether the players can, or want to, enforce it if agreement is not reached. In the classical example of employer-union wage negotiations the union can call out a strike if it is not satisfied with the wage offered by the employer. But how long should the strike last? What will be its consequences? Will all workers join? Are there perhaps different and better ways to put pressure on management? Also, which outcome can the employer enforce, if any, in case no agreement is reached?

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In this paper, following Vartiainen (2007), we assume that the disagreement outcome is determined endogenously, namely by the bargaining solution. Specifically, the bargaining solution assigns a *pair* of outcomes, namely a compromise outcome and a disagreement outcome. The possible interpretations of such a bargaining solution are parallel to the usual interpretations of a classical bargaining solution in the situation where the disagreement outcome is exogenous. From a positive point of view, a *classical* bargaining solution predicts or describes the compromise outcome, i.e., it tells us what this outcome is *given* that the players reach agreement. From this point of view, a bargaining solution in the situation *without* exogenous disagreement outcome predicts both the compromise outcome for the case that the players reach an agreement, and the disagreement outcome in the opposite case. From a normative point of view, a *classical* bargaining solution functions like an outside arbitrator and proposes a compromise outcome, but this outcome should be 'reasonable' given the exogenous disagreement outcome. In the situation *without* exogenous disagreement outcome, a bargaining solution proposes a compromise as well as a disagreement outcome, such that the compromise is 'reasonable' when compared to the disagreement outcome. We will refine and detail these interpretations in Section 2 below.

Within this framework, Vartiainen (2007) proposes and axiomatically characterizes a bargaining solution which extends the classical Nash bargaining solution for bargaining problems with fixed, exogenous disagreement point. That solution maximizes the Nash product, i.e., the product of the gains of the players from the compromise outcome over the disagreement outcome.

By contrast, the solution proposed in our paper depends explicitly on the utopia point and extends the solution of Raiffa (1953) and Kalai-Smorodinsky (1975) for classical bargaining problems. This extension works as follows. First, the assigned compromise point is indeed the classical Kalai-Smorodinsky (KS) outcome for the assigned disagreement outcome. That is, it is the Pareto optimal point on the straight line joining this disagreement outcome and its associated utopia point. Second, the assigned disagreement outcome is the point on the straight line joining the assigned compromise point and the associated 'anti-utopia point', obtained by taking the minimum utilities of the players below the compromise point; it is, thus, a 'converse' KS outcome. The main original condition justifying the classical Kalai-Smorodinsky solution is individual monotonicity: it implies that if the utopia point stays fixed, then the players should benefit from increased availability of favorable outcomes. In defining the KS solution for the case where the disagreement outcome is not exogenous, we thus apply the same logic also to the determination of the disagreement outcome: given that the anti-utopia point does not change, the players should suffer from the increased availability of unfavorable outcomes.

We present two axiomatic characterizations of this solution. In the first one, the crucial axiom is called Independence of Non-Utopia information (INU). This condition is relatively strong and, under an additional condition, says that the compromise and disagreement outcomes in two different problems should be the same if the associated utopia and anti-utopia points coincide. In the second characterization, INU is replaced by three much weaker axioms, including a monotonicity condition.

Another extension of the Kalai-Smorodinsky solution to bargaining problems without fixed disagreement point is proposed in Vartiainen (2002), but this solution is quite different from our extension.[1]

The framework in our paper and in Vartiainen (2007) has resemblance to the one in Thomson (1981), who also considers bargaining problems without disagreement point. Thus, a bargaining problem is defined merely as a utility-possibility set. Thomson introduces the notion of *reference point* as a *function* of the bargaining problem.[2] The key difference to the classical disagreement point is that no player can unilaterally enforce the reference point. It may thus serve, rather, as a hypothetical outcome to which the players compare proposals made during negotiations. The key difference with our (and Vartiainen's) approach is that we assume that also the reference point (disagreement outcome) is determined by the solution.

In situations where an arbitrator, or a mediator, makes choices for the players (cf. Luce and Raiffa, 1957), the reference point may also result from a noncooperative, strategic game between the players, and the arbitrator (bargaining solution) assigns a compromise point based on the reference point. Effectively, this way a noncooperative game is turned into a strictly competitive game which may have a value, comparable to a zero-sum game. Such arbitration games have received renewed attention recently, see Kalai and Kalai (2010).

In Section 2 we present a more detailed discussion of bargaining with endogenous disagreement. In Section 3 we formally introduce the extended Kalai-Smorodinsky solution, show that it is non-empty valued and characterize the domain of bargaining problems for which it is single-valued. In Section 4 we present two axiomatic characterizations of the solution on domains where it is single-valued. We also show that the axioms in these characterizations are independent.

All proofs are collected in the Appendix.

Notation For $x, y \in \mathbb{R}^2$, $x > y$ means $x_i > y_i$ and $x \geqslant y$ means $x_i \geqslant y_i$ for $i = 1, 2$. Similarly for $<$ and $\leqslant$. By $[x, y]$ we denote the line segment with endpoints x and y. The cardinality of a set $X \subseteq \mathbb{R}^2$ is denoted by $|X|$. For $a, x \in \mathbb{R}^2$, $ax := (a_1x_1, a_2x_2)$, $aX := \{ax \mid x \in X\}$, and $a+X := \{a+x \mid x \in X\}$. The set $(-1, -1)X$ is also denoted by $-X$. By $\mathbb{R}^2_+$ we denote the (strictly) positive quadrant of $\mathbb{R}^2$. By $\mathrm{conv}(X)$ we denote the convex hull of the set X.

[1] It assigns the points of intersection of the straight line connecting the *global* utopia and anti-utopia points with the boundary of the feasible set and, thus, extends the Kalai-Rosenthal (Kalai and Rosenthal, 1978) solution rather than the Kalai-Smorodinsky solution.

[2] Herrero (1998) considers endogenous reference points in so-called bargaining problems with claims. Also these reference points are a function of the bargaining problem and, in this case, the claims point.

2 Bargaining with endogenous disagreement

A *bargaining problem* U is a compact and convex subset of $\mathbb{R}^2$ such that $x > y$ for some $x, y \in U$. Elements of U are called *outcomes* and represent the utilities of two players. By $\mathcal{U}$ we denote the set of all bargaining problems.

A *classical bargaining problem* is a pair (U, d), where $U \in \mathcal{U}$ and $d \in U$; the outcome d is called the *disagreement outcome*, and it results if the players do not reach agreement. By $\mathcal{B}$ we denote the set of all classical bargaining problems. A *classical bargaining solution* is a map $F : \mathcal{B} \to \mathbb{R}^2$ with $F(U, d) \in U$ for all $(U, d) \in \mathcal{B}$.

In contrast, a *bargaining solution* or, briefly, a *solution* is a correspondence $f : \mathcal{U} \to \mathbb{R}^2 \times \mathbb{R}^2$ such that $s, r \in U$ and $s \neq r$ for all $U \in \mathcal{U}$ and $(s, r) \in f(U)$. For a pair $(s, r) \in f(U)$, we call s the *compromise outcome* and r the *disagreement outcome*.

We now discuss how solutions with endogenous disagreement can be interpreted, also offering some perspectives that go beyond Vartiainen (2007).[3] Classical bargaining theory commonly distinguishes between positive interpretations, according to which bargaining solutions aim to predict the outcome of a bargaining process, and normative interpretations, according to which solutions express a judgment of what outcome would be normatively 'best' or 'fairest' and should therefore be proposed by an arbitrator or mediator if such a person is appointed. Since our extended bargaining solutions return two outcomes – a compromise outcome s and a disagreement outcome r – we may classify potential interpretations according to which of the two outcomes are interpreted positively ('players') and which normatively ('mediator'). This yields four possible interpretations overall, see Table 1.

		disagreement r	
		players	mediator
compromise s	players	*Case 1*	*Case 3*
	mediator	*Case 2*	*Case 4*

Table 1 Four potential interpretations of bargaining solutions

We discuss these cases in turn. In the 'doubly positive' Case 1, s is the predicted compromise outcome of bargaining (without arbitration or mediation), and r the predicted outcome failing agreement. Outcome r plays the role of players' mental reference point, representing their common beliefs of what would happen failing agreement. Both s and r are predicted to emerge as the result of the bargaining process, in the course of which various proposals and threats might have been on the table.[4]

Cases 2, 3 and 4 represent three variants of how a mediator could intervene in the bargaining process. These variants are not merely hypothetical but can be

[3] This discussion has benefitted from helpful comments of an anonymous referee.

[4] Case 1 is perhaps closest to the idea of a reference point as in Thomson (1981).

observed in practice. For instance, they correspond to different forms of so-called *Alternative Dispute Resolution* (ADR). Especially in Anglo-Saxon common law systems, ADR has become a wide-spread practice aimed to avoid costly and lengthy court trials through reaching a compromise beforehand.[5] While ADR always assigns a central role to a so-called mediator, the exact nature of this role differs across different forms of ADR. Bargaining theory with endogenous disagreement allows one to study some existing forms of ADR in virtue of the interpretations of Cases 2, 3 and 4. To understand why this is so, two pieces of background information are worth noting. Firstly, the role of the mediator in ADR does typically *not* consist in elaborating a binding compromise. Instead, any compromise needs both parties' approval. Should this compromise have been proposed by the mediator, this proposal was non-binding. This marks a key difference between ADR and orthodox forms of dispute resolution such as *court trials* and *arbitration*; there, the role of the judge resp. arbitrator is precisely to dictate a binding compromise.[6] Secondly, prior to entering ADR both parties have contractually agreed to the mediator's precise role, whatever this role consists in. So, parties cannot later withdraw from the ADR procedure, and any threats or incentives placed by the mediator are credible in the game-theoretic sense. Now we turn to the specific Cases 2, 3 and 4.

In Case 2, the mediator proposes a non-binding compromise s (after listening to both parties, i.e., 'learning' the bargaining problem U at hand). This makes s salient and externally approved. If both players accept s, it is implemented. If the parties do not both accept the proposal and do not reach an alternative compromise, the non-cooperative outcome r is predicted. So, r once again operates as a reference point or 'threat', creating an incentive to accept the proposal s (as long as $s > r$).

In Case 4, the mediator not just proposes a non-binding compromise, but also underpins this proposal with the threat of forcing a 'bad' binding outcome r on the parties (typically including sanctions or fines) which takes effect in the eventuality that the players neither agree to s nor manage to reach an alternative compromise. This of course presupposes that players have contractually authorized the mediator to dictate a binding disagreement outcome (which players may plausibly do to facilitate a compromise). Once the mediator has announced r, players effectively face an exogenous disagreement outcome. While classical bargaining theory can be used to model bargaining *given* the mediator's announced r, we also address how r is determined.

Case 3 gives more responsibility to the parties: the mediator does not propose a compromise to the parties but mediates between them to help them find a compromise by themselves. Just as in Case 4, to create an incentive to compro-

[5]The United Kingdom legislation strongly encourages, if not *de facto* forces, parties to engage in an ADR process prior to meeting before court (since the 2004 judgment in the *Halsey* landmark case). For general introductions to ADR, see for instance Lynch (2001) and Blake et al. (2011).

[6]The difference between a court trial and arbitration is that the former is instituted by the state, whereas the latter is based on a contractual agreement between both parties to submit to an arbitration procedure.

mise, the mediator imposes a binding outcome r (typically including sanctions or fines) that takes effect if no compromise is reached.

We ultimately leave it to the reader which interpretation to prefer and which applications to focus on. As with bargaining theory in general, the theory with endogenous disagreement captures its intended applications only in a stylized and simplified way. For instance, the model abstracts away certain goals of ADR, such as the goal of inducing a change and ideally a convergence of the parties' preferences. We hope that the connection to ADR will motivate future research and generalizations.

3 Bargaining problems and the extended Kalai-Smorodinsky solution

In this paper we focus on a particular solution, which extends the classical Kalai-Smorodinsky bargaining solution (Raiffa, 1953; Kalai and Smorodinsky, 1975). For a bargaining problem $U \in \mathcal{U}$, the *Pareto optimal set* is the set

$$P(U) := \{x \in U \mid \text{for all } y \in U,\, y \geqslant x \text{ implies } y = x\}$$

and the *anti-Pareto optimal set* is the set

$$AP(U) := \{x \in U \mid \text{for all } y \in U,\, y \leqslant x \text{ implies } y = x\}.$$

The classical Kalai-Smorodinsky bargaining solution assigns to each classical bargaining problem (U, d) the unique point $KS(U, d) \in P(U)$ on the straight line through d and the *utopia point*

$$u(U, d) = \left(\max_{x \in U,\ x \geqslant d} x_1, \min_{x \in U,\ x \geqslant d} x_2 \right).$$

The *extended Kalai-Smorodinsky solution* is the correspondence $k : \mathcal{U} \to \mathbb{R}^2 \times \mathbb{R}^2$ defined by

$$(s, r) \in k(U) \Leftrightarrow s = KS(U, r),\ r = -KS(-U, -s) \text{ and } s \neq r$$

for all $U \in \mathcal{U}$ and $s, r \in U$. Thus, $(s, r) \in k(U)$ exactly if the following three conditions are satisfied: (i) $s \neq r$; (ii) s is the classical Kalai-Smorodinsky outcome when r is viewed as the disagreement outcome; and (iii) r results similarly from s when we reverse the problem or, equivalently, r is the unique point in $AP(U)$ on the straight line through s and the *anti-utopia point*

$$a(U, s) := \left(\min_{x \in U,\ x \leqslant s} x_1, \min_{x \in U,\ x \leqslant s} x_2 \right).$$

See Figure 1 for an illustration.

Our first result is that the extended Kalai-Smorodinsky solution is non-empty valued. The proof is based on an elementary fixed point argument, slightly complicated by the fact that the Pareto and anti-Pareto optimal sets of a bargaining problem U may have one or both endpoints in common. Clearly, in that case, by definition of k – in particular the condition $s \neq r$ – such an endpoint cannot be the solution outcome.

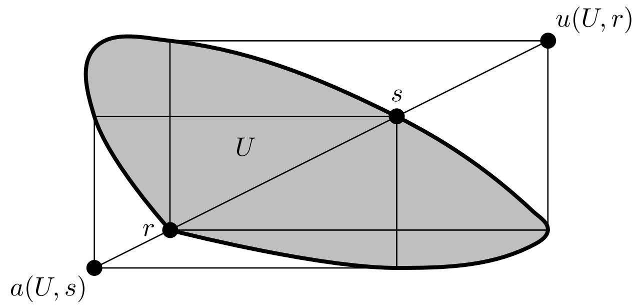

Figure 1: An illustration of the extended Kalai-Smorodinsky solution

Theorem 3.1 *$k(U) \neq \emptyset$ for all $U \in \mathcal{U}$.*

We note that k does not have to assign a *unique* pair of outcomes to a bargaining problem. For instance, let U be the convex hull of the points $(6, 0)$, $(8, 0)$, $(0, 6)$, and $(0, 8)$. Then it is not difficult to check that

$$k(U) = \{((s_1, s_2), (r_1, r_2)) \mid 2 \leqslant s_1 \leqslant 6,\ r_1 = s_1 - 1,\ s_1 + s_2 = 8,\ r_1 + r_2 = 6\}.$$

In this example the Pareto optimal and anti-Pareto optimal sets are parallel line segments. In fact, a sufficient but not necessary condition for k to assign a unique pair of outcomes to a problem U is that $P(U)$ and $AP(U)$ do *not* contain parallel line segments. Theorem 3.2 below provides an exact description of the class of all bargaining problems on which k is unique. We first introduce some additional terminology.

For $x \neq y$ and $x' \neq y'$ the line segments $[x, y]$ and $[x', y']$ are *parallel* if the straight lines ℓ and ℓ' containing these line segments are parallel. In that case, the *vertical distance* between $[x, y]$ and $[x', y']$ is the number $v = |z_2 - z'_2|$ for (any) $z \in \ell$ and $z' \in \ell'$ with $z_1 = z'_1$; v is infinite if ℓ and ℓ' are vertical. Similarly, the *horizontal distance* between $[x, y]$ and $[x', y']$ is the number $h = |z_1 - z'_1|$ for (any) $z \in \ell$ and $z' \in \ell'$ with $z_2 = z'_2$; h is infinite if ℓ and ℓ' are horizontal.

Now let $\mathcal{D}_k$ denote the set of bargaining problems U with $|k(U)| = 1$.

Theorem 3.2 *Let $U \in \mathcal{U}$. Then $U \in \mathcal{D}_k$ if and only if there are* no *parallel line segments $[\bar{x}, \underline{x}] \subseteq AP(U)]$ and $[\bar{y}, \underline{y}] \subseteq P(U)$ with $\bar{x}_1 < \underline{x}_1$ and $\bar{y}_1 < \underline{y}_1$ and such that the vertical distance v and horizontal distance h between these line segments satisfy the following conditions:*

(i) $\frac{1}{2}v = \bar{y}_2 - \bar{x}_2 = \underline{y}_2 - \underline{x}_2$,

(ii) the lengths[7] *of $[\bar{x}, \underline{x}]$ and $[\bar{y}, \underline{y}]$ both exceed $\sqrt{h^2 + v^2}$.*

See Figure 3 in the Appendix for an illustration. The theorem implies that we do not lose much generality if we restrict attention to domains of bargaining

[7] The length of a line segment is the Euclidean distance between its endpoints.

problems within $\mathcal{D}_k$. We conclude this section with a remark, listing some domains on which k is single-valued.

Remark 3.3 Theorem 3.2 implies that the extended Kalai-Smorodinsky solution k is single-valued on each of the following domains:

(a) $\{U \in \mathcal{U} \mid U$ is strictly convex$\}$.

(b) $\{U \in \mathcal{U} \mid AP(U)$ or $P(U)$ contains no line segment$\}$.

(c) $\{U \in \mathcal{U} \mid$ no line segments $S \subseteq AP(U)$ and $S' \subseteq P(U)$ are parallel$\}$.

Clearly, the domain in (a) is a subset of the domain in (b), which in turn is a subset of the domain in (c).

4 Two axiomatic characterizations of the extended Kalai-Smorodinsky solution

In this section we give two axiomatic characterizations of the extended Kalai-Smorodinsky solution k on domains on which k is single-valued. In each characterization all axioms except one are basic and shared with the extended Nash solution. In the first characterization the additional axiom is an informational constraint (Independence of Non-Utopia Information), while in the second it is a monotonicity property analogous to such properties used in characterizations of the classical Kalai-Smorodinsky solution.

We formulate our conditions for a solution f defined on a domain $\mathcal{D} \subseteq \mathcal{U}$ with $|f(U)| = 1$ for all $U \in \mathcal{D}$. Instead of $f(U) = \{(s,r)\}$ we write $f(U) = (s,r)$ and regard f as a function rather than a correspondence.

A bargaining problem $U' \in \mathcal{U}$ is a *positive affine transformation* of a bargaining problem $U \in \mathcal{U}$ if there are $a \in \mathbb{R}^2_+$ and $b \in \mathbb{R}^2$ such that $U' = aU + b$. A bargaining problem $U \in \mathcal{U}$ is *symmetric* if $(x_1, x_2) \in U \Leftrightarrow (x_2, x_1) \in U$ for all $x \in \mathbb{R}^2$.

The first condition is an extended version of the usual Pareto optimality condition.

Extended Pareto Optimality (EPO): For each $U \in \mathcal{D}$, $f(U) \in P(U) \times AP(U)$.

In particular from a normative view point it is natural to require Pareto optimality of the compromise outcome. Requiring anti-Pareto optimality of the disagreement outcome reflects that we wish this outcome to be as severe as possible in order to induce acceptance of the compromise outcome.[8]

The following two conditions are standard in classical axiomatic bargaining theory. They have similar justifications in the present model.

[8] Disagreement poses a threat to the players only if $s > r$ (where $f(U) = (s,r)$), as a referee rightly noticed. In our two characterization results, EPO could be weakened by restricting the anti-Pareto optimality requirement on r to those cases in which $s > r$.

Symmetry (SYM): For each symmetric $U \in \mathcal{D}$, if $f(U) = (s, r)$ then $s_1 = s_2$ and $r_1 = r_2$.

Scale Invariance (SI): For all $U \in \mathcal{D}$ and $a \in \mathbb{R}^2_+$, $b \in \mathbb{R}^2$ with $aU + b \in \mathcal{D}$, if $f(U) = (s, r)$ then $f(aU + b) = (as + b, ar + b)$.

We now turn to axioms used in only one of our two characterizations. The first characterization is based on an informational restriction which extends and modifies similar conditions used in characterizations of the classical Kalai-Smorodinsky solution.

Independence of Non-Utopia Information (INU): For all $U, V \in \mathcal{D}$, if $f(V) = (s, r) \in P(U) \times AP(U)$, $u(U, r) = u(V, r)$ and $a(U, s) = a(V, s)$, then $f(U) = (s, r)$.

This condition says that if $f(V) = (s, r)$ and we consider a problem U such that s and r are Pareto and anti-Pareto optimal in U and also the associated utopia and anti-utopia points do not change, then the solution does not change: $f(U) = (s, r)$ as well.

Our second characterization replaces INU by three other axioms, each of which seems normatively defensible and extends classical axioms. The first of these axioms requires that the compromise outcome weakly Pareto dominates the disagreement outcome, i.e., that the disagreement outcome is a threat to both players.

Pareto Dominance (PD): For every $U \in \mathcal{D}$, if $f(U) = (s, r)$ then $s \geqslant r$.

The next condition requires that the outcome for any bargaining problem U be unchanged if one removes possible alternatives x from U that are extreme in the sense of giving some individual even less utility than under the original disagreement outcome while giving the other individual even more utility than under the original compromise outcome.

Independence of Extreme Alternatives (IEA): For all $U, U' \in \mathcal{D}$, writing $(s, r) = f(U)$, if $U' \subseteq U$ and for every $x \in U \backslash U'$ there is an agent i such that $x_i < r_i$ and $x_j > s_j$ for $j \neq i$, then $f(U') = f(U)$.

This condition is a weak version of the condition of IIA (Independence of Irrelevant Alternatives, extended to endogenous disagreement), which underlies the extended Nash bargaining solution. IEA relaxes IIA by restricting it to the case that two sets U and U' differ only in 'extreme' alternatives.

The final condition is a variant of classical monotonicity conditions. It is well-known from classical bargaining theory that plausible bargaining solutions usually satisfy *some but not any* kind of monotonicity property. Our monotonicity condition requires that if additional alternatives become available, then, at least under certain extra conditions, the compromise outcome improves weakly and the disagreement outcome worsens weakly for each player. Roughly speaking, the justification is that additional possibilities should give room for better compromise outcomes but also worse disagreement outcomes. In order to formulate the axiom we define, for a bargaining problem $U \in \mathcal{D}$, the *global utopia point* and the *global anti-utopia point* by

$$u(U) = \left(\max_{x \in U} x_1, \max_{x \in U} x_2 \right), \quad a(U) = \left(\min_{x \in U} x_1, \min_{x \in U} x_2 \right).$$

Restricted Monotonicity (RM): For all $U, U' \in \mathcal{D}$, writing $(s, r) = f(U)$ and $(s', r') = f(U')$, if $U \subseteq U'$, $u(U') = u(U, r)$, and $a(U') = a(U, s)$, then $s' \geq s$ and $r' \leq r$.

Clearly, the conditions on the utopia and global utopia points and the anti-utopia and global anti-utopia points considerably restrict this monotonicity condition.[9]

The domain $\mathcal{D}$ is *closed under truncation* if whenever it contains U then it also contains every bargaining problem of the form $\{x \in U \mid \alpha \leq x_i \leq \beta\}$ for some $i \in \{1, 2\}$ and $\alpha, \beta \in \mathbb{R}$ with $a_i(U) \leqslant \alpha < \beta \leqslant u_i(U)$. The domain $\mathcal{D}$ is *minimally rich* if it is closed under truncation and contains all polytopes in $\mathcal{D}_k$.[10] For instance, the whole domain $\mathcal{D}_k$ and the (small) domain of all polytopes in $\mathcal{D}_k$ are both minimally rich by Theorem 3.2.

Theorem 4.1 *Let $\mathcal{D} \subseteq \mathcal{D}_k$ be minimally rich and let $f : \mathcal{D} \to \mathbb{R}^2 \times \mathbb{R}^2$ be a solution satisfying $|f(U)| = 1$ for all $U \in \mathcal{D}$. Then the following statements are equivalent:*

(a) *f is the extended Kalai-Smorodinsky solution on $\mathcal{D}$.*

(b) *f satisfies EPO, SYM, SI, and INU.*

(c) *f satisfies EPO, SYM, SI, PD, IEA, and RM.*

The characterizations in Theorem 4.1 are tight. We show this by means of examples of solutions defined on a minimally rich domain $\mathcal{D} \subseteq \mathcal{D}_k$. Proofs are left to the reader. We start with demonstrating tightness of the six axioms in characterization (c).

(1) For each $U \in \mathcal{D}$, write $(s, r) := k(U)$ and let $f^1(U) := (t, r)$, where t is the point in $[r, s]$ which is closest to s subject to U containing at least one of the points $(t_1, a_2(U, s))$ and $(a_1(U, s), t_2)$ (note that possibly $t = s$). Then f^1 satisfies SYM, SI, PD, IEA, and RM, but not EPO.

(2) Define the solution f^2 in the same way as k but now based on a non-symmetric version of the KS-solution (cf. Peters and Tijs, 1985). Such a solution satisfies EPO, SI, PD, IEA, and RM, but not SYM.

[9] Note that the antecedent in RM implies that $u(U) = u(U, r)$ and $a(U) = a(U, s)$.
[10] A polytope is the convex hull of finitely many points in $\mathbb{R}^2$.

(3) Let T be the convex hull of $(0,0)$, $(4,0)$, and $(0,2)$. Define the solution f^3 as follows. For all $U \in \mathcal{D}$ with $T \subseteq U$, $a(U) = (0,0)$, and $u(U) = (4,2)$, define $f^3(U)$ as $(s,(0,0))$ where s is the point of intersection of $P(U)$ with the line segment $[(3,\frac{1}{2}),(4,2)]$. Otherwise, define $f^3(U) = k(U)$. Then f^3 satisfies EPO, SYM, PD, IEA, and RM, but not SI.

(4) For each $U \in \mathcal{U}$, let $(s(U), r(U)) := k(U)$, let $\hat{s}(U)$ resp. $\hat{r}(U)$ be the element of U with first coordinate $r_1(U)$ resp. $s_1(U)$ and with maximal resp. minimal second coordinate, denote the set of non-extreme outcomes relative to k by $\overline{U} = \{x \in U \mid x_i \leqslant s_i(U) \text{ or } x_j \geqslant r_i(U) \text{ for all distinct } i, j\}$, and call $U \in \mathcal{U}$ *essentially symmetric* if some positive affine transformation of $\overline{U}$ is symmetric. For all $U \in \mathcal{D}$, define $f^4(U)$ as $(\hat{s}(U), \hat{r}(U))$ if $[\hat{s}(U) \in P(U)$, $\hat{r}(U) \in AP(U)$ and U is not essentially symmetric], and as $k(U)$ otherwise. Then f^4 satisfies EPO, SYM, SI, IEA, and RM, but not PD.

(5) For all $U \in \mathcal{D}$, define $f^5(U)$ as (s,r) where s $[r]$ is the intersection of $P(U)$ $[AP(U)]$ with the line segment joining the global utopia point and the global anti-utopia point of U. Then f^5 satisfies EPO, SYM, SI, PD, and RM, but not IEA.

(6) Let T be the convex hull of $(0,0)$, $(4,0)$, $(2,1)$, and $(0,1)$. For all $U \in \mathcal{D}$, define $f^6(U)$ as (s,b) if $U = aT + b$ for some $a \in \mathbb{R}^2_+$, $b \in \mathbb{R}^2$, where s is the Nash bargaining solution of (U,b), and as $k(U)$ otherwise. Then f^6 satisfies EPO, SYM, SI, PD, and IEA, but not RM.

Next, we show that the axioms in characterization (b) are tight.

(7) The solution f^1 satisfies SYM, SI, and INU, but not EPO.

(8) The solution f^2 satisfies EPO, SI, and INU, but not SYM.

(9) Let T be as in (3) and define the solution f^7 by $f^7(T) = ((3,\frac{1}{2}),(0,0))$, and by $f^7(U) = k(U)$ for all $U \in \mathcal{D}_k$ with $U \neq T$. Then f^7 satisfies EPO, SYM, INU, but not SI.

(10) The solutions f^4, f^5, and f^6 all satisfy EPO, SYM, and SI, but not INU.

We conclude with a few remarks.

Remark 4.2 A partial characterization of the extended Kalai-Smorodinsky solution on the whole domain $\mathcal{U}$ is provided in Valkengoed (2006)[11], at the expense of rather technical conditions.

Remark 4.3 Variations on the characterization of k can be obtained by imposing different conditions of 'minimal richness'. For instance, Theorem 4.1 would still hold – with some modifications of the proof – on some subdomains of $\mathcal{D}_k$ that contain all strictly convex bargaining problems.

[11] Master thesis, supervised by the third author of this paper.

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A Appendix: proofs

Proof of Theorem 3.1. Let $U \in \mathcal{U}$. Then $AP(U)$ is the graph of a strictly decreasing convex function g on an interval $[\alpha, \beta]$ with $(\alpha, g(\alpha))$ and $(\beta, g(\beta))$ the points of $AP(U)$ with minimal and maximal first coordinates, respectively. If $\alpha = \beta$ (so that $AP(U)$ consists of a unique outcome) then $\{(KS(U, (\alpha, g(\alpha))), (\alpha, g(\alpha)))\} = k(U)$ and we are done. From now on we assume $\alpha < \beta$. Define the function $\varphi : [\alpha, \beta] \to [\alpha, \beta]$ by $\varphi(\gamma) = -KS_1(-U, -KS(U, (\gamma, g(\gamma))))$. Observe that if $\varphi(\gamma^*) = \gamma^*$ for some $\gamma^* \in [\alpha, \beta]$ and $KS(U, (\gamma^*, g(\gamma^*))) \neq (\gamma^*, g(\gamma^*))$ then $(KS(U, (\gamma^*, g(\gamma^*))), (\gamma^*, g(\gamma^*))) \in k(U)$.

Of course, $\varphi(\alpha) \geqslant \alpha$ and $\varphi(\beta) \leqslant \beta$. Suppose that $(\alpha, g(\alpha)) \in P(U)$. Then $\varphi(\alpha) = \alpha$, but $(KS(U, (\alpha, g(\alpha))), (\alpha, g(\alpha))) \notin k(U)$ since $KS(U, (\alpha, g(\alpha))) = (\alpha, g(\alpha))$. Below, however, we will prove:

$$\text{There is an } \varepsilon_1 > 0 \text{ with } \varphi(\gamma) > \gamma \text{ for all } \gamma \in (\alpha, \alpha + \varepsilon_1]. \tag{1}$$

Similarly, if $(\beta, g(\beta)) \in P(U)$ we have:

$$\text{There is an } \varepsilon_2 > 0 \text{ with } \varphi(\gamma) < \gamma \text{ for all } \gamma \in [\beta - \varepsilon_2, \beta). \tag{2}$$

Clearly, we can then take ε_1 and ε_2 in (1) and (2) such that $\alpha + \varepsilon_1 < \beta - \varepsilon_2$. Now define the interval $[\alpha', \beta']$ by $\alpha' = \alpha$ if $(\alpha, g(\alpha)) \notin P(U)$ and $\alpha' = \alpha + \varepsilon_1$ if $(\alpha, g(\alpha)) \in P(U)$, and $\beta' = \beta$ if $(\beta, g(\beta)) \notin P(U)$ and $\beta' = \beta - \varepsilon_2$ if $(\beta, g(\beta)) \in P(U)$. Then, since φ is continuous, the intermediate value theorem implies that in all cases there is a point $\gamma^* \in [\alpha', \beta']$ with $\varphi(\gamma^*) = \gamma^*$ and $KS(U, (\gamma^*, g(\gamma^*))) \neq (\gamma^*, g(\gamma^*))$ and, thus, $k(U) \neq \emptyset$.

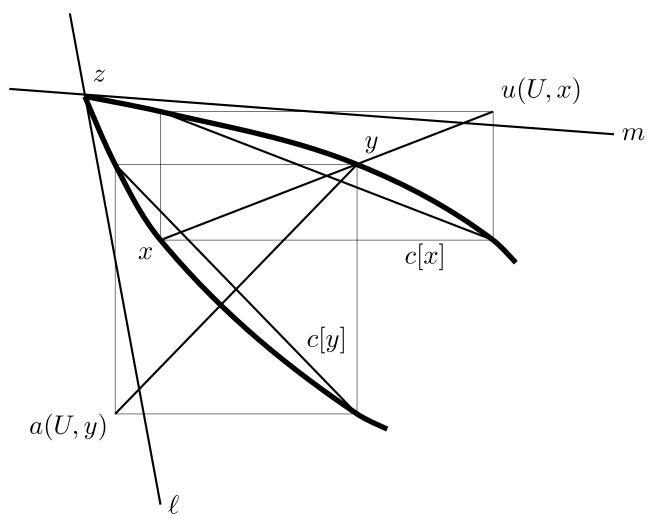

Figure 2: Illustrating the proof of (1)

We are left to prove (1) and (2). We only show (1), the proof of (2) is analogous. So suppose $z := (\alpha, g(\alpha)) \in P(U)$. See Figure 2 for an illustration of the remainder of the proof.

Let m and ℓ be the supporting lines of U at z as drawn in Figure 2. (That is, m is the limit of supporting lines at $P(U)$ and ℓ is the limit of supporting lines at $AP(U)$.) Let μ be the absolute value of the slope of m and let λ be the absolute value of the slope of ℓ. Then $\lambda > \mu$.

For $x \in AP(U) \setminus P(U)$ let $\sigma(x)$ denote the slope of the straight line through x and $u(U, x)$. Let $c[x]$ denote the line segment with endpoints $(x_1, u_2(U, x))$ and $(u_1(U, x), x_2)$. Then $\sigma(x)$ is equal to the absolute value of the slope of $c[x]$. Therefore, $\sigma(x)$ converges to μ if $x \in AP(U)$ converges to z.

For $y \in P(U) \setminus AP(U)$ let $\tau(y)$ denote the slope of the straight line through y and $a(U, y)$. Let $c[y]$ denote the line segment with endpoints $(y_1, a_2(U, y))$ and $(a_1(U, y), y_2)$. Then $\tau(y)$ is equal to the absolute value of the slope of $c[y]$. Therefore, $\tau(y)$ converges to λ if $y \in P(U)$ converges to z.

We conclude that $\tau(y) > \sigma(x)$ for $y \in P(U)$ and $x \in AP(U)$ close to z. This implies the existence of an ε_1 as in (1). □

Proof of Theorem 3.2

For the only-if part, suppose that $[\bar{x}, \underline{x}] \subseteq AP(U)]$ and $[\bar{y}, \underline{y}] \subseteq P(U)$ are as in the Theorem. Let $\bar{r} \in [\bar{x}, \underline{x}]$ with $\bar{r}_1 = \bar{y}_1$ and $\underline{r} \in [\bar{x}, \underline{x}]$ with $\underline{r}_2 = \underline{y}_2$. For each $r \in [\bar{r}, \underline{r}]$ let $s(r) \in [\bar{y}, \underline{y}]$ with $s(r)_2 - r_2 = v/2$. Then it is straightforward to check that $(s(r), r) \in k(U)$ for each $r \in [\bar{r}, \underline{r}]$. Thus, $U \notin \mathcal{D}_k$. See Figure 3 for an illustration.

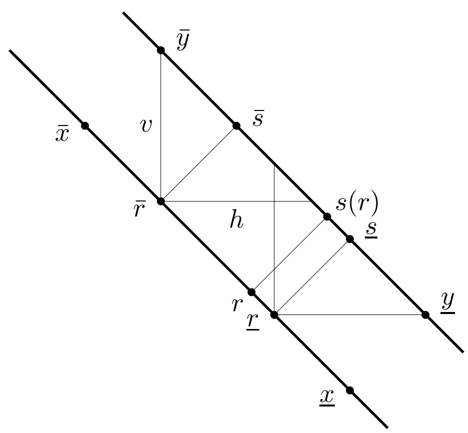

Figure 3: Illustrating the proof of Theorem 3.2

We now prove the if-part. Assume $U \notin \mathcal{D}_k$, i.e. $|k(U)| > 1$. We will construct $[\bar{x}, \underline{x}] \subseteq AP(U)]$ and $[\bar{y}, \underline{y}] \subseteq P(U)$ as in the theorem.

For any $x \in AP(U)$ let $\sigma(x)$ denote the slope of the straight line through x and $u(U, x)$ (as in the proof of Theorem 3.1). Since $\sigma(x)$ is equal to the absolute value of the slope of the line segment $c[x]$ connecting the points $(x_1, u_2(U, x))$ and $(u_1(U, x), x_2)$, and the absolute values of these slopes weakly increase if x_1 increases – the line segments $c[x]$ are chords of the weakly decreasing concave function the graph of which contains the Pareto optimal set of U – we have that $\sigma(x)$ weakly increases if x_1 increases. $(*)$

Similarly, for any $y \in P(U)$ let $\tau(y)$ denote the slope of the straight line through y and $a(U,y)$ (again as in the proof of Theorem 3.1). Then by an analogous argument $\tau(y)$ weakly increases if y_1 decreases. $(**)$

Let $(\bar{s},\bar{r})$ and $(\underline{s},\underline{r})$ be the elements of $k(U)$ with maximal and minimal second coordinates, respectively. By definition of k we have $\tau(s) = \sigma(r)$ for all $(s,r) \in k(U)$. Therefore, by $(*)$ and $(**)$ we must have $\sigma(x) = \tau(y)$ for all $x \in AP(U)$ with $\bar{r}_1 \leqslant x_1 \leqslant \underline{r}_1$ and all $y \in P(U)$ with $\bar{s}_1 \leqslant y_1 \leqslant \underline{s}_1$. In particular, $\sigma(x)$ is constant for $\bar{r}_1 \leqslant x_1 \leqslant \underline{r}_1$, which implies that the line segments $c[x]$ for $x \in [\bar{r},\underline{r}]$ are parallel; but this means that they must be on the same straight line m through $\bar{s}$ and $\underline{s}$. Let $\bar{y}$ be the upper endpoint of $c[\bar{r}]$ and let $\underline{y}$ be the lower endpoint of $c[\underline{r}]$. Then $[\bar{y},\underline{y}] \subseteq P(U)$, $\bar{y}_1 = \bar{r}_1$ and $\underline{y}_2 = \underline{r}_2$. See, again, Figure 3 for an illustration. Similarly, let ℓ be the straight line through $\bar{r}$ and $\underline{r}$, then $[\bar{x},\underline{x}] \subseteq AP(U)$, where $\bar{x}$ is the point of ℓ with $\bar{x}_2 = \bar{s}_2$ and $\underline{x}$ is the point of m with $\underline{x}_1 = \underline{s}_1$. Now it is straightforward to check that $[\bar{x},\underline{x}]$ and $[\bar{y},\underline{y}]$ satisfy the conditions in the theorem. □

Proof of Theorem 4.1.

(1) We first prove that (a) implies (b) and (c). We leave verification of EPO, SYM, SI, INU, PD, and IEA of k on $\mathcal{D}$ to the reader. To show RM, consider $U,U' \in \mathcal{D}$ satisfying the antecedent of RM, i.e., $U \subseteq U'$, $u(U') = u(U,r)$, and $a(U') = a(U,s)$, where $(s,r) := k(U)$. Since $r < s$ and $U \subseteq U'$, there are unique points $\bar{r} < \bar{s}$ in the intersection of the line through r and s with the boundary of U'. We show that $(\bar{s},\bar{r}) = k(U')$, which completes the proof of RM since, clearly, $\bar{r} \leq r$ and $\bar{s} \geq s$. Note that $u(U',\bar{r}) \leq u(U') = u(U,r) \leq u(U',\bar{r})$, so that $u(U',\bar{r}) = u(U,r)$, whence $KS(U',\bar{r}) = \bar{s}$. Similarly, $a(U',\bar{s}) \geq a(U') = a(U,s) \geq a(U',\bar{s})$, so that $a(U',\bar{s}) = a(U,s)$, whence $KS(-U',-\bar{s}) = -\bar{r}$. It follows that $k(U') = (\bar{s},\bar{r})$.

(2) We now prove that (b) implies (a). Suppose f satisfies the four conditions in (b) and let $U \in \mathcal{D}$. We have to prove that $f(U) = k(U)$. Let $k(U) = (s,r) \in P(U) \times AP(U)$. Then $s > r$ (this follows from the requirement that there must be $x,y \in U$ with $x > y$). Let V be the convex hull of the six points s, r, $(s_1, a_2(U,s))$, $(a_1(U,s), s_2)$, $(u_1(U,r), r_2)$, and $(r_1, u_2(U,r))$. We will prove that $V \in \mathcal{D}$ and $f(V) = (s,r)$. This will conclude the proof of (b) $\Rightarrow$ (a), since by INU, $f(V) = (s,r)$ implies $f(U) = (s,r)$ and, thus, $f(U) = k(U)$.

Consider the positive affine transformation

$$(x_1,x_2) \mapsto (\varphi_1(x_1), \varphi_2(x_2)) := \left(\frac{x_1 - r_1}{s_1 - r_1}, \frac{x_2 - r_2}{s_2 - r_2}\right)$$

which maps r to $(0,0)$, s to $(1,1)$, and V to some set V'. Then V' is the convex hull of the set

$$\{(0,0),(1,1),(1,\tfrac{a_2(U,s)-r_2}{s_2-r_2}),(\tfrac{a_1(U,s)-r_1}{s_1-r_1},1),(\tfrac{u_1(U,r)-r_1}{s_1-r_1},0),(0,\tfrac{u_2(U,r)-r_2}{s_2-r_2})\}\ .$$

Note that

$$\frac{a_2(U,s)-r_2}{s_2-r_2}=\frac{a_1(U,s)-r_1}{s_1-r_1} \text{ and } \frac{u_2(U,r)-r_2}{s_2-r_2}=\frac{u_1(U,r)-r_1}{s_1-r_1}\ .$$

Thus, V' is a symmetric polytope, and it is sufficient to prove that $V' \in \mathcal{D}_k$: for this implies $V \in \mathcal{D}$ by minimal richness of $\mathcal{D}$; and by SYM and EPO, we have $f(V') = ((1,1),(0,0))$ and thus, by SI, $f(V) = (s,r)$.

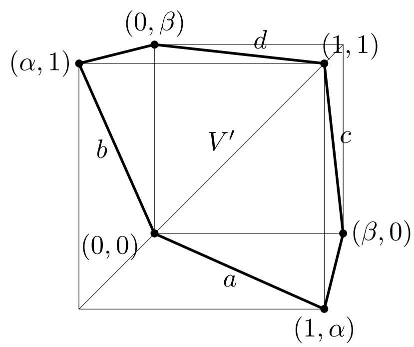

Figure 4: Illustrating the proof of part (2) of Theorem 4.1. The letters a,b,c,d denote line segments, $\alpha = \varphi_1(a_1(U,s)) = \varphi_2(a_2(U,s))$, and $\beta = \varphi_1(u_1(U,r)) = \varphi_2(u_2(U,r))$

We are left to prove that $V' \in \mathcal{D}_k$, i.e., that $|k(V')| = 1$. Consider Figure 4 with notations as there. For $k(V')$ to be non-unique there are, in view of Theorem 3.2, two possible cases to examine: (1) a is parallel to d and (2) a is parallel to c. (The cases involving b are analogous.) In case (1) we must have $\beta = 1-\alpha > 1$. Denote the vertical and horizontal distances between a and d by v and h, then the length of a is equal to $\sqrt{1+\alpha^2}$ whereas $\sqrt{v^2+h^2} > \sqrt{\beta^2+\beta^2} > \sqrt{1+\alpha^2}$, so that a does not satisfy condition (ii) in Theorem 3.2. In case (2) we must have $\beta = 2$ and $\alpha = -1$. In particular, $AP(V')$ is the line segment $[(-1,1),(1,-1)]$ and $P(V')$ is the line segment $[(0,2),(2,0)]$, so that again condition (ii) in Theorem 3.2 is violated.

(3) We finally prove that (c) implies (a). Suppose f satisfies the six conditions in (b) and let $U \in \mathcal{D}$. Let $(s,r) := f(U)$. We proceed in several steps.

Claim 1. $s > r$.

To prove this claim, assume the contrary. As $s \geqslant r$ by PD and $s \neq r$ by definition of a bargaining solution, we may assume $s_1 = r_1$ and $s_2 > r_2$ (the other case is analogous).

Consider first the truncated set $\widehat{U} = \{x \in U \mid x_2 \leqslant s_2\}$, which is in $\mathcal{D}$ by minimal richness. Note that for each $x \in U \setminus \widehat{U}$ we have $x_2 > s_2$ and hence $x_1 < s_1 = r_1$ as $s \in P(U)$; so by IEA, $f(\widehat{U}) = f(U)$. Next consider the set $T = \{x \in \widehat{U} \mid x_2 \geqslant r_2\}$, which is again in $\mathcal{D}$ by minimal richness. Next, note that for each $x \in \widehat{U} \setminus T$ we have $x_2 < r_2$ and hence $x_1 > r_1 = s_1$ as $r \in AP(\widehat{U})$; so again by IEA, $f(T) = f(\widehat{U})$. Altogether we have $f(T) = f(U)$ where $T = \{x \in U \mid r_2 \leqslant x_2 \leqslant s_2\}$. Moreover, by construction of T, we have $u(T, r) = u(T)$ and $a(T, s) = a(T)$.

As $T \supsetneq [s, r]$, we have $u(T) \neq s$ or $a(T) \neq r$. Suppose $u(T) \neq s$ (the proof is analogous if $a(T) \neq r$). Then, since $s \in P(T)$, $u(T) \notin T$. Choose $\alpha, \beta \in \mathbb{R}$ with $a_2(T) < \alpha < u_2(T)$ and $a_1(T) < \beta < u_1(T)$ such that the set

$$T' = \text{conv}\{a(T), (a_1(T), u_2(T)), (u_1(T), a_2(T)), (u_1(T), \alpha), (\beta, u_2(T))\}$$

is a positive affine transformation of a symmetric polytope and such that $T \subseteq T'$. Then $T' \in \mathcal{D}$ as $\mathcal{D}$ is minimally rich and $T' \in \mathcal{D}_k$ by Theorem 3.2. Let $(s', r') = f(T')$. As T' is symmetric up to a positive affine transformation, SI and SYM imply that $r', s' \in T' \cap [a(T'), u(T')]$. So, as $u(T') \notin T'$, we have $s' < u(T') = u(T)$, whence in particular $s'_2 < u_2(T)$. On the other hand, since $T \subseteq T'$, $u(T') = u(T) = u(T, r)$ and $a(T') = a(T) = a(T, s)$, we have by RM that $s' \geqslant s$, so that $s'_2 = u_2(T)$. This contradiction completes the proof of Claim 1.

Claim 2. Let $U' = \{x \in U \mid a(U, s) \leqslant x \leqslant u(U, r)\}$. Then $U' \in \mathcal{D}$ and $f(U') = (s, r)$.

To prove this, first observe that, since $s > r$ by Claim 1, U' arises from U by a double truncation. Hence, $U' \in \mathcal{D}$. We next prove that all outcomes in $U \setminus U'$ are extreme alternatives in the sense of IEA. Suppose that $x \in U$ with $x_1 < r_1$. Suppose $x_2 \leqslant s_2$. Since $r < s$, we have $x < s$, hence $x \geqslant a(U, s)$. Also, $x < u(U, r)$ since $s \leqslant u(U, r)$. Thus, $x \in U'$. Hence, if $x \in U \setminus U'$ then $x_1 < r_1$ implies $x_2 > s_2$. Similarly, $x \in U \setminus U'$ and $x_2 < r_2$ imply $x_1 > s_1$. Suppose now $x \in U$ and $x \geqslant r$. Then $x \leqslant u(U, r)$ and since $r < s$, whence $r \geqslant a(U, s)$, we have $x \geqslant a(U, s)$, so that $x \in U'$. Altogether we have proved that the antecedent of IEA holds for $U' \subseteq U$, so that $f(U') = (s, r)$.

In view of Claim 2 and the definition of k it is sufficient to prove that $f(U') = k(U')$. In view of SI of f and k we may assume that $a(U', s) = (0, 0)$ and $u(U', r) = (1, 1)$. Denote $L = [(0, 0), (1, 1)]$ and $U^0 = \{x \in \mathbb{R}^2 \mid (0, 0) \leqslant x \leqslant (1, 1)\}$. If $r, s \in L$ then clearly $k(U') = (s, r) = f(U')$ and we are done. Otherwise, without loss of generality $s \notin L$. We proceed by choosing $\hat{s}, \hat{r} \in L$ as follows. If $r \notin L$ then choose $\hat{s}, \hat{r}$ such that: (i) $\hat{s} \ngeqslant s$, $\hat{r} \nleqslant r$; (ii) there is a line ℓ through $\hat{s}$ intersecting the boundary of U^0 at points $(\alpha, 1)$ and $(1, \beta)$ such that $\hat{r}_1 < \alpha < 1$, $\hat{r}_2 < \beta < 1$ and such that the set U' is weakly below ℓ; (iii) there is a line m, not parallel to ℓ, through $\hat{r}$ intersecting the boundary of U^0 at points $(0, \gamma)$ and $(\delta, 0)$ such that $0 < \gamma < \hat{s}_2$, $0 < \delta < \hat{s}_1$ and such that the set U' is weakly above m.[12] See Figure 5 for an illustration. If $r \in L$ then we still

[12]The first candidates for $\hat{s}$ and $\hat{r}$ are the points s' and r', where s' is the point of intersection of L and $P(U')$ and r' is the point of intersection of L and $AP(U')$; and for ℓ and m the lines through s' and r' supporting U'. If those lines happen to be parallel, or if one or more of the numbers α, β, γ, and δ do not satisfy the desired constraints, one can take $\hat{s} = s' + (\varepsilon, \varepsilon)$ and/or $\hat{r} = r' - (\varepsilon', \varepsilon')$, where $0 < \varepsilon, \varepsilon'$ are sufficiently small, and shift up and if necessary slightly rotate ℓ and/or m.

choose $\hat{s}$ as above but set $\hat{r} = (0, 0)$, and $\gamma = \delta = 0$. Let V be the polytope with vertices $(0, 1)$, $(1, 0)$, $(\alpha, 1)$, $(1, \beta)$, $(0, \gamma)$, and $(\delta, 0)$. Then $V \in \mathcal{D}_k$ by Theorem 3.2 and therefore $V \in \mathcal{D}$ since $\mathcal{D}$ is minimally rich.

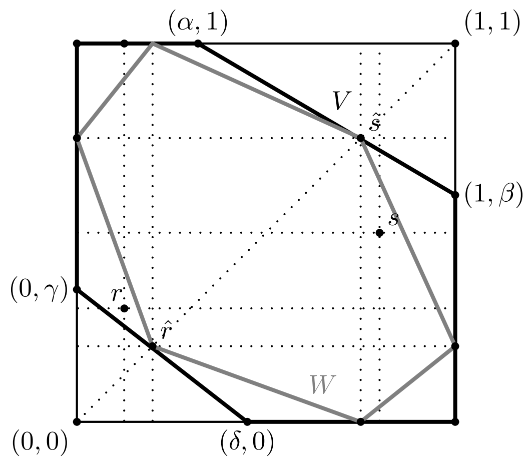

Figure 5: Illustrating the proof of part (3) of Theorem 4.1. The black curve is the boundary of the set V and the gray curve is the boundary of the set W.

Claim 3. $f(V) \in \{x \in V : x \geq s\} \times \{x \in V : x \leq r\}$.
The claim follows from RM and Claim 2, noting that $U' \subseteq V$ and that $u(V) = u(U', r)$ $(= (1, 1))$ and $a(V) = a(U', s)$ $(= (0, 0))$.

Let W be the convex hull of the points $\hat{s}$, $\hat{r}$, $(\hat{s}_1, 0)$, $(1, \hat{r}_2)$, $(0, \hat{s}_2)$, and $(\hat{r}_1, 1)$.

Claim 4. $W \in \mathcal{D}$ and $f(W) = (\hat{s}, \hat{r})$.
That $W \in \mathcal{D}$, in particular that $W \in \mathcal{D}_k$, follows by the same argument as used in the last part of (2) above. Since W is symmetric, EPO and SYM imply $f(W) = (\hat{s}, \hat{r})$.

Claim 5. $f(V) = (\hat{s}, \hat{r})$.
To prove this, we note that by construction of V and W we have $(1, 1) = u(V) = u(W, \hat{r})$ and $(0, 0) = a(V) = a(W, \hat{s})$. Since $W \subseteq V$, the claim follows from RM and Claim 4.

We can now complete the proof of part (3) and the theorem. Claim 5, Claim 3, and the definition of $\hat{s}$ and $\hat{r}$ imply that we must have $s, r \in L$ since otherwise we obtain a contradiction. But in that case we have $(s, r) = k(U') = f(U')$. $\square$